

# Math 218A Final Exam (Take Home)

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1. Let  $(E, B, \pi)$  be a mapping triple. If both  $E$  and  $B$  are compact Hausdorff spaces and  $\pi$  is a surjective local homeomorphism, then  $(E, B, \pi)$  is a covering space.

Let  $y \in B$ , and let  $A_y = \pi^{-1}(y)$ . We first argue that  $\#(A)$  is finite.

$\{y\}$  is a closed set, and  $\pi$  is a local homeomorphism (thus it at least locally bi-continuous), so  $A_y$  is closed, and thus compact in  $E$  (as  $A \subseteq E$ ,  $E$  compact).

$\pi$  is a local homeomorphism, so for each  $x \in A_y$  there is an open neighborhood  $\mathcal{O}_x$  such that  $\pi|_{\mathcal{O}_x}$  is a homeomorphism.

$\bigcup_{x \in A_y} \mathcal{O}_x$  is an open covering for  $A_y$ , so there is a finite subcover, which we'll call  $\mathcal{O}_1, \dots, \mathcal{O}_n$ . Each  $\mathcal{O}_j$  can contain at most one  $x \in A_y$ , as if  $x, \bar{x} \in A_y$  with  $x, \bar{x} \in \mathcal{O}_j$ , then  $\pi|_{\mathcal{O}_j}(x) = \pi|_{\mathcal{O}_j}(\bar{x})$ , thus  $x = \bar{x}$  (as  $\pi|_{\mathcal{O}_j}$  is a homeomorphism and thus bijective).

So, there are at most  $n$  values in  $A_y$ , so  $A_y$  is finite.

Write  $A_y = \{x_1, \dots, x_n\}$ .  $E$  is Hausdorff, so we can separate any two distinct points using open sets. Thus, for each  $x_j$  there are open sets separating  $x_j$  from  $x_i$  (where  $i \neq j$ ). Select such sets; call them  $U_i^j$  and  $U_j^i$ , where  $x_i \in U_i^j$  and  $x_j \in U_j^i$ . WLOG, these open sets can be thought of as subsets of the corresponding  $\mathcal{O}_i$  or  $\mathcal{O}_j$  respectively (as if not, just replace  $U_i^j$  with  $U_i^j \cap \mathcal{O}_i$  and  $U_j^i$  with  $U_j^i \cap \mathcal{O}_j$ ). By construction, we thus have  $U_i^j \cap U_j^i = \emptyset$  as these are (derived from) the open separations guaranteed by  $E$ 's Hausdorff-ness.

Let  $U_i = \bigcap_{j=1, j \neq i}^n U_i^j$ .  $U_i$  is a non-empty ( $y_i \in U_i$ ) open set that is contained within  $\mathcal{O}_i$ . Further, each  $U_i \cap U_j = \emptyset$  for  $i \neq j$ . Take  $V = \bigcap_{i=1}^n \pi(U_i)$ , which is a non-empty ( $y \in \pi(U_i)$  for each  $i$ ) open in  $B$  (as  $\pi|_{U_i}$  is a homeomorphism, thus bi-continuous).

$V$  is a local open neighborhood of  $y$  that is evenly covered. It is covered by the disjoint open neighborhoods  $\hat{U}_j = U_j \cap \pi^{-1}(V)$ , where each  $\hat{U}_j$  is an open neighborhood of  $x_j$ .

Thus,  $(E, B, \pi)$  is a covering space.

- 2) Let  $X$  and  $Y$  be metric spaces with  $X$  compact.  $f_n : X \rightarrow Y$  is a sequence of continuous functions that converge pointwise to a function  $f$ . Show that the convergence is uniform if and only if  $\{f_n\}_{n=1}^{\infty}$  is a uniformly equicontinuous subset of  $C(X, Y)$ .

Assume that  $f_n \rightarrow f$  uniformly.

$f$  is continuous as each  $f_n$  is continuous and the convergence is uniform.  $f$  has a compact domain, so  $f$  is uniformly continuous, as is each  $f_n$ . Let  $\epsilon > 0$  be given. Let  $x_1, x_2 \in X$ . By the triangle inequality:

$$\rho_Y(f_n(x_1), f_n(x_2)) \leq \rho_Y(f_n(x_1), f(x_1)) + \rho_Y(f(x_1), f(x_2)) + \rho_Y(f(x_2), f_n(x_2))$$

Intuitively, the first and third terms can be bounded as the convergence is assumed to be uniform and the second can be bounded as  $f$  is uniformly continuous.

More formally.  $f_n \rightarrow f$  uniformly so there is an  $N \in \mathbb{N}$  such that  $n > N$  implies that  $\rho_Y(f(x), f_n(x)) < \frac{\epsilon}{3}$  for any  $x \in X$ .

$f$  is uniformly continuous so there is a  $\delta_0 > 0$  so that  $\rho_X(x_1, x_2) < \delta_0$  implies that  $\rho_Y(f(x_1), f(x_2)) < \frac{\epsilon}{3}$ .

Further, each  $f_j$  is uniformly continuous, so for this  $\epsilon$  there is a  $\delta_j$  such that  $\rho_X(x_1, x_2) < \delta_j$  then  $\rho_Y(f_j(x_1), f_j(x_2)) < \epsilon$ . Take  $\delta = \min_{j=0 \text{ to } N} \delta_j$ .

For this selection of  $\delta$ , if  $\rho_X(x_1, x_2) < \delta$  then  $\rho_Y(f_j(x_1), f_j(x_2)) < \epsilon$ , so  $\{f_n\}_{n=1}^{\infty}$  is uniformly equicontinuous.

Assume that  $\{f_n\}_{n=1}^{\infty}$  is uniformly equicontinuous.

Claim: For all  $x \in X$ ,  $S \subseteq \{f_n\}_{n=1}^{\infty}$   $S(x)$  is totally bounded.

Let  $\epsilon > 0$  be given.  $f_n \rightarrow f$  pointwise, so for this  $x$  there is an  $N \in \mathbb{N}$  so that  $n > N$  we have  $\rho_Y(f(x), f_n(x)) < \epsilon$ , so  $\{f_n\}_{n=N+1}^{\infty}(x) \subseteq B_{\epsilon}(f(x), Y)$ . Thus  $\{B_{\epsilon}(f_j(x), Y)\}_{j=1}^N$  and  $B_{\epsilon}(f(x), Y)$  cover  $\{f_n\}_{n=1}^{\infty}(x)$ , so  $S(x)$  is totally bounded.

$\{f_n\}_{n=1}^{\infty}$  is totally bounded by Ascoli-Arzelà, thus  $\{f_n\}_{n=1}^{\infty}$  has a subsequence that uniformly converges to  $f$ , thus  $f$  is continuous.  $f$  is continuous with a compact domain, thus  $f$  is uniformly continuous.

$$\rho_Y(f_n(x_1), f(x_1)) \leq \rho_Y(f_n(x_1), f_n(a)) + \rho_Y(f_n(a), f(a)) + \rho_Y(f(a), f(x_1))$$

For all  $x, a \in X$ , let  $\delta_2$  be the  $\delta$  provided by uniform equicontinuity for  $\frac{\epsilon}{3}$ .

For the first term: if  $\rho_X(x_1, a) < \delta_2$  then  $\rho_Y(f_n(a), f_n(x_1)) < \frac{\epsilon}{3}$  by uniform equicontinuity.

For the third term: for all  $x, a \in X$ ,  $f$  is uniformly continuous so there exists a  $\delta_1 > 0$  so that if  $\rho_X(a, x_1) < \delta_1$  then  $\rho_Y(f(a), f(x_1)) < \frac{\epsilon}{3}$ .

For the second term:  $f_n \rightarrow f$  pointwise, so for this fixed  $a$  there is a  $N_a$  so that  $n > N_a$  then  $\rho_Y(f_n(a), f(a)) < \frac{\epsilon}{3}$ .

Let  $\delta = \min\{\delta_1, \delta_2\}$ .

For each  $a \in X$ ,  $x_1 \in B_{\delta}(a, X)$  with  $n > N_a$  we have the above inequality.  $X = \bigcup_{a \in X} B_{\delta}(a, X)$  so these form an open covering for  $X$ .  $X$  is compact, so there is a finite subcover  $B_{\delta}(a_1, X), \dots, B_{\delta}(a_m, X)$  with corresponding  $N_1, \dots, N_m$ .

Take  $N = \max N_j$ . Now the above inequality applies for all  $x \in X$  whenever  $n > N$ , so  $f_n \rightarrow f$  uniformly.

3) Show that the intersection of a nested sequence of closed, connected subsets of a Hausdorff space,  $X$ , is connected.

Let  $C_1 \supseteq C_2 \supseteq \dots$  be the nested sequence of closed connected subsets of  $X$ . First note that each  $C_i$  is a closed subset of a compact space, and is thus compact. If  $C_i = \emptyset$  for some  $i$  then we see that this is trivially true, so WLOG assume that  $C_i \neq \emptyset$  for all  $i$ .

Let  $C = \bigcap_{i=1}^{\infty} C_i$ .  $C \neq \emptyset$  and is compact, as this family of sets satisfies the finite intersection property.

Assume that  $C$  was not connected, thus there is an open  $U$  and  $V$  such that  $U \cup V = C$ ,  $U \cap V = \emptyset$  and  $U$  and  $V$  are both non-empty.

Examine  $A_j = C_j \setminus (U \cup V)$ .  $A_j \neq \emptyset$  as if it were, then  $C_j$  would not be connected.  $A_j$  is closed, thus compact, so  $A = \bigcap_{j=1}^{\infty} A_j$  is non-empty. But  $A$  is present in each  $C_j$ , so  $A \subset C \setminus (U \cup V) = \emptyset$ , a contradiction.

So  $C$  is connected.

- 4) Show that if  $X$  and  $Y$  are compact Hausdorff spaces with  $f : X \rightarrow Y$  and  $gr(f)$  is closed in  $X \times Y$  then  $f$  is continuous.

We first show that  $\pi_X : X \times Y \rightarrow X$  (the projection map to  $X$ ) is a closed map.

$\pi_X$  is continuous under the product topology. Let  $A \subseteq X \times Y$  be closed, thus compact (as both  $X$  and  $Y$  are compact, and  $X \times Y$  is compact by the finite analog to Tychonoff's theorem). Thus  $\pi_X(A)$  is compact, and thus closed in  $X$ . So,  $\pi_X$  is a closed map.

Now, let  $x_0 \in X$ , and let  $V$  be a neighborhood of  $f(x_0)$ . We want to show that  $f^{-1}(V)$  is open.

$Y - V$  is closed in  $Y$ , so  $X \times (Y \setminus V)$  is closed in the product topology.  $gr(f)$  is closed, so  $gr(f) \cap (X \times (Y \setminus V))$  is closed.

$\pi_X$  is a closed map, so  $B = \pi_X (gr(f) \cap (X \times (Y \setminus V)))$  is closed. Thus  $Y \setminus B$  is open in  $Y$ . Now note that  $Y \setminus B = f^{-1}(V)$ , so  $f^{-1}(V)$  is open, thus  $f$  is continuous at  $x_0$ . The choice of  $x_0$  was arbitrary, so  $f$  is continuous on all of  $X$ .

- 5) (a) Show that a space  $X$  is Hausdorff if and only if the diagonal

$$\Delta := \{(x, x) : x \in X\} \subset X \times X$$

is closed in  $X \times X$ .

Assume  $X$  is Hausdorff. Let  $(x, y) \in X \times X$  such that  $x \neq y$ .  $X$  is Hausdorff so there exists open sets  $\mathcal{O}_x$  and  $\mathcal{O}_y$  separating  $x$  and  $y$  in  $X$ , with  $x \in \mathcal{O}_x$ ,  $y \in \mathcal{O}_y$  and  $\mathcal{O}_x \cap \mathcal{O}_y = \emptyset$ .  $\mathcal{O}_x \times \mathcal{O}_y$  is open in  $X \times Y$ .  $\mathcal{O}_x$  and  $\mathcal{O}_y$  are disjoint, so  $(\mathcal{O}_x \times \mathcal{O}_y) \cap \Delta = \emptyset$ , so  $(X \times X) \setminus \Delta$  is open, so  $\Delta$  is closed.

Assume that  $\Delta$  is closed. Let  $x, y \in X$  such that  $x \neq y$ .  $(x, y) \notin \Delta$  so  $(x, y) \in (X \times X) \setminus \Delta$  (which is open), so  $(x, y)$  is in  $N$ , an open neighborhood of  $(x, y)$  that is disjoint from  $\Delta$ .

$N$  is open in  $X \times X$  so  $N = \bigcup_{\alpha} (U_{\alpha} \times V_{\alpha})$  with all  $U_{\alpha}$  and  $V_{\alpha}$  open in  $X$ . In each case,  $U_{\alpha} \cap V_{\alpha} = \emptyset$  for if not then  $N$  would not be disjoint from  $\Delta$ . So, for some  $\alpha$  there are open sets  $U_{\alpha}$  and  $V_{\alpha}$  so that  $x \in U_{\alpha}$  and  $y \in V_{\alpha}$ ; if not, then  $(x, y)$  would not be in  $N$ . The sets  $U_{\alpha}$  and  $V_{\alpha}$  are open sets separating  $x$  and  $y$ , so  $X$  is Hausdorff.

- (b) Let  $\sim$  be an equivalence relation in a space  $X$ .  $\pi : X \rightarrow (X/\sim)$  is an open mapping. Show that the quotient space  $(X/\sim)$  with the quotient topology is Hausdorff if and only if  $\sim$  is closed in  $X \times X$ .

Assume  $(X/\sim)$  with the quotient topology is Hausdorff.

Note that  $(\pi \times \pi) : X \times X \rightarrow (X/\sim) \times (X/\sim)$  is a continuous map, where  $(\pi \times \pi)(x, y) = (\pi(x), \pi(y))$ .

$(x, y) \in \Delta$  if and only if  $(x, y) \in \sim$ . By (a),  $\Delta$  is closed, so  $(\pi \times \pi)^{-1}(\Delta) = \sim$  is closed in  $X \times X$ .

Assume  $\sim$  is closed in  $(X \times X)$ . By the vital extra hypothesis that  $\pi$  is an open map, we find that  $\pi \times \pi$  is an open map.

If we define the set  $A = \{(x, y) \in X \times X : (x, y) \notin \sim\} = (X \times X) \setminus \sim$  is open in  $X \times X$ .

$(\pi \times \pi)(A)$  is open in  $(X/\sim) \times (X/\sim)$ , so  $(\pi \times \pi)(A) = (X/\sim) \times (X/\sim) \setminus \Delta$  is open, so  $\Delta$  is closed, so  $X/\sim$  is Hausdorff.

- (c) What is the connection between (a) and (b)?

The obvious answer is "(a) was used to prove (b) in both directions!". In addition, note that all equivalence relations are reflexive, so  $\Delta$  necessarily includes  $(x, x)$  for all  $x \in X$ .

- 6) (a)  $A$  is a retract of  $X$ ,  $a \in A$ . If a loop  $\sigma \in \Lambda(A, a)$  represents a non-trivial element in  $\Pi_1(A, a)$ , then  $\sigma$  represents a non-trivial element in  $\Pi_1(X, a)$ .

Let  $r$  be the retract of  $X$  to  $A$ .

First note that if  $\gamma_1$  and  $\gamma_2$  are homotopic paths in  $X$ , then  $f \circ \gamma_1$  is homotopic to  $f \circ \gamma_2$ , as we can simply apply  $f$  to the homotopy.

So, if  $[\sigma]$  was a trivial element in  $\Pi_1(X, a)$  then  $r \circ \sigma$  is trivial in  $\Pi_1(A, a)$ , but  $r \circ \sigma = \sigma$  as every point in the loop  $\sigma$  is in  $A$ , and  $r$  fixes  $A$ , so  $\sigma$  would be a trivial element in  $\Pi_1(A, a)$ , a contradiction. So a non-trivial element of  $\Pi_1(A, a)$  is also non-trivial element of  $\Pi_1(X, a)$ .

- (b) Prove that the closed disk  $D^2$  cannot be retracted onto its boundary  $S^1$ .

Assume that there is some retract of  $D^2$  to  $S^1$ ,  $r$ .

Let  $x \in S^1$ .  $\Pi_1(S^1, x)$  is non-trivial, so there is some non-trivial  $\sigma \in \Lambda(S^1, x)$ .  $\Pi_1(D^2, x)$  is trivial.

$\sigma$  trivial in  $\Pi_1(D^2, x)$ , so  $r \circ \sigma = \sigma$  is trivial in  $S^1$ , which is a contradiction.

So there is no retract of  $D^2$  to  $S^1$ .

- (c) Show that any continuous map  $f$  of the 2-disk into itself must have a fixed point.

Assume that  $f$  had no fixed point. Define a function  $r(x)$  as the point on  $S^1$  that intersects the ray from  $f(x)$  to  $x$ .

Formally, let's consider  $R^2$  as a vector space under the standard set of vector operations. Define  $l_x(t) = f(x) + (x - f(x))t = f(x)(1 - t) + x(t)$ . Note that this defines the line between  $x$  and  $f(x)$  (which is well defined, as we have assumed that  $x \neq f(x)$  for all  $x \in D^2$ ). We will restrict  $t \in (0, \infty)$ . Note that  $l_x(0) = f(x)$  and  $l_x(1) = x$ .

We wish to verify that the point of intersection is well defined. To this end, we want to find  $t$  such that  $\|l_x(t)\| = 1$ .

$$\begin{aligned} 1 &= \|l_x(t)\| \\ &= \|f(x) + (x - f(x))t\| \\ &= (f(x) + (x - f(x))t) \cdot (f(x) + (x - f(x))t) \\ &= \|f(x)\|^2 + 2tf(x) \cdot x - 2\|f(x)\|^2 t + \|x - f(x)\|^2 t^2 \end{aligned}$$

Rearranging this into a quadratic equation in  $t$ , we see:

$$0 = (\|f(x)\|^2 - 1) + 2t(f(x) \cdot x - \|f(x)\|^2) + \|x - f(x)\|^2 t^2$$

The quadratic formula gives us two solutions, one strictly positive, and the other non-positive. As a consequence of the map we desire, we discard the non-positive solutions, leaving only

$$t = \frac{(\|f(x)\|^2 - f(x) \cdot x) + 2\sqrt{(f(x) \cdot x - \|f(x)\|^2)^2 - \|x - f(x)\|^2 (\|f(x)\|^2 - 1)}}{2\|x - f(x)\|^2}$$

Again, note that this is well defined as  $x \neq f(x)$  by hypothesis.

This is a ratio of two functions, both continuous in  $x$ , and we are not dividing by 0, so  $t$  varies continuously with  $x$ .

Taking this value for  $t$ , now note that  $l_x(t)$  is also continuous in  $x$ . Define  $r(x) = l_x(t)$  to be this continuous function.

Now, note that by the map's construction, if  $x$  is chosen on  $S^1$ , we have  $r(x) = x$ , so  $r(x)$  is a continuous function that fixes  $S^1$ .

Thus we have defined a retract from  $D^2$  to  $S^1$ , a contradiction by part (b), so this map must fail to be well defined for some  $x \in D^2$ , so  $f$  must have a fixed point on  $D^2$ .