Solving Cauchy-Euler ODEs by Substitution

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If you have a second order Cauchy-Euler differential equation

$$a_2 x^2 y'' + a_1 x y' + a_0 y = g(x)$$

you can convert this equation into a linear ODE with constant coefficients by making a substitution: $x(t) = e^t$. We can undo this substitution using $t(x) = \ln x$. It's clear how to use this substitution to get rid of the x terms, but what of the derivatives?

We start by noting that this differential equation is all in terms of x:

$$a_2 x^2 \frac{d^2 y}{dx^2}(x) + a_1 x \frac{dy}{dx}(x) + a_0 y(x) = g(x)$$

We want to convert our differential equation so that it is in terms of t rather than x. To do this, let's define a new version of the solution:

$$\tilde{y}(t) = y(x(t)) = y(e^t)$$

Our new differential equation will be in terms of $\tilde{y}(t)$ and its derivatives (which will be with respect to t). With this new version of the function, we can represent our original function:

$$y(x) = \tilde{y}(t(x)) = \tilde{y}(\ln x)$$

We can use these new functions to find out what $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$ in terms of $\tilde{y}(t)$ and its derivatives:

$$\frac{dy}{dx} = \frac{d}{dx}\left[y(x)\right] = \frac{d}{dx}\left[\tilde{y}(\ln x)\right] = \frac{d\tilde{y}}{dt}\frac{1}{x} = e^{-t}\frac{d\tilde{y}}{dt}$$

(this is an application of the chain rule)

We use this first derivative to calculate the second derivative:

$$\begin{aligned}
\frac{d^2 y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{1}{x}\frac{d\tilde{y}}{dt}\right] &= \frac{d}{dx}\left[\frac{1}{x}\tilde{y}'(\ln x)\right] \\
&= \frac{d}{dx}\left[\frac{1}{x}\right]\tilde{y}'(\ln x) + \frac{1}{x}\frac{d}{dx}\left[\tilde{y}'(\ln x)\right] \\
&= -\frac{1}{x^2}\frac{d\tilde{y}}{dt} + \frac{1}{x^2}\frac{d^2\tilde{y}}{dt^2} \\
&= e^{-2t}\left(\frac{d^2\tilde{y}}{dt^2} - \frac{d\tilde{y}}{dt}\right)
\end{aligned}$$

For higher order equations, you can continue to apply this same process to find the higher derivatives.

Summarizing:

$$\frac{dy}{dx} = e^{-t}\frac{d\tilde{y}}{dt}$$

and

$$\frac{d^2y}{dx^2} = e^{-2t}\left(\frac{d^2\tilde{y}}{dt^2} - \frac{d\tilde{y}}{dt}\right)$$

Notice that these didn't depend on the particular differential equation; these work any time that you use this substitution.

In the abstract, we can apply this directly to the general form:

$$a_2x^2\frac{d^2y}{dx^2} + a_1x\frac{dy}{dx} + a_0y = g(x) \xrightarrow{x=e^t} a_2e^{2t}\left[e^{-2t}\left(\frac{d^2\tilde{y}}{dt^2} - \frac{d\tilde{y}}{dt}\right)\right] + a_1e^t\left[\frac{d\tilde{y}}{dt}e^{-t}\right] + a_0y\left(e^t\right) = g\left(e^t\right)$$

$$\rightarrow a_2\frac{d^2\tilde{y}}{dt^2} + \left(a_1 - a_2\right)\frac{d\tilde{y}}{dt} + a_0\tilde{y} = g\left(e^t\right) \rightarrow a_2\tilde{y}'' + \left(a_1 - a_2\right)\tilde{y}' + a_0\tilde{y} = g\left(e^t\right)$$

So, one way to approach this class of problem is just to remember that the substitution yields:

$$a_2x^2y'' + a_1xy' + a_0y = g(x) \xrightarrow{x=e^t} a_2\tilde{y}'' + \left(a_1 - a_2\right)\tilde{y}' + a_0\tilde{y} = g\left(e^t\right)$$

This ODE now has constant coefficients, and can thus be approached by our standard methods. Once we have a solution, $\tilde{y}(t)$, we can apply the reverse substitution to get a solution for our original ODE:

$$y(x) = \tilde{y}(\ln x)$$

Section 4.7, problem 33

As an example, look at the differential equation:

$$x^2 y'' + 10xy' + 8y = x^2$$

Now, substitute into the differential equation (using what we found above!):

$$e^{2t}\left[e^{-2t}\left(\frac{d^2\tilde{y}}{dt^2} - \frac{d\tilde{y}}{dt}\right)\right] + 10e^{t}\left(\frac{d\tilde{y}}{dt}e^{-t}\right) + 8\tilde{y}(t) = e^{2t}$$

Simplifying:

$$\tilde{y}'' + 9\tilde{y}' + 8\tilde{y} = e^{2t}$$

Alternately, note that $a_2x^2y'' + a_1xy' + a_0y = g(x) \xrightarrow{x=e^t} a_2\tilde{y}'' + (a_1 - a_2)\tilde{y}' + a_0\tilde{y} = g(e^t)$, so

$$x^2 y'' + 10xy' + 8y = x^2 \xrightarrow{x=e^t} \tilde{y}'' + 9\tilde{y}' + 8\tilde{y} = e^{2t}$$

At this point, we can solve using the standard methods. The differential operator here is

$$L = D^2 + 9D + 8$$

The auxiliary equation for the homogeneous case is $m^2 + 9m + 8 = (m+1)(m+8) = 0$ so

$$\tilde{y}_c(t) = c_1e^{-t} + c_2e^{-8t}$$

We guess that a particular solution could have the form $\tilde{y}_p(t) = Ae^{2t}$. Applying the differential operator:

$$L\tilde{y}_p = 4Ae^{2t} + 18Ae^{2t} + 8Ae^{2t} = 30Ae^{2t}$$

This is supposed to equal to e^{2t}, so $A = \frac{1}{30}$, resulting in our particular solution

$$\tilde{y}_p(t) = \frac{1}{30}e^{2t}$$

$\tilde{y}(x) = \tilde{y}_c(x) + \tilde{y}_p(x)$ so our solution is $\tilde{y}(t) = c_1e^{-t} + c_2e^{-8t} + \frac{1}{30}e^{2t}$.

Now, reverse the substitution:

$$y(x) = \tilde{y}(\ln x) = c_1e^{-\ln x} + c_2e^{-8\ln x} + \frac{1}{30}e^{2\ln x}$$

or:

$$y(x) = c_1\frac{1}{x} + c_2\frac{1}{x^8} + \frac{1}{30}x^2$$