3a. Find a non-constant morphism $\mathbb{P}^1 \to \mathbb{P}^2$

The map

$$[x_0 : x_1] \mapsto [x_0 : x_1 : x_0 + x_1]$$

is a morphism on between $\mathbb{P}^1$ and $\mathbb{P}^2$, as it involves only homogeneous functions of degree 1, and is clearly non-constant.

7. (a) Show that a space X is Hausdorff if and only if the diagonal

$$\Delta(X) := \{(x, x) : x \in X\} \subset X \times X$$

is closed in $X \times X$.

Assume X is Hausdorff. Let $(x, y) \in X \times X$ such that $x \neq y$. X is Hausdorff so there exists open sets O_x and O_y separating x and y in X, with $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \emptyset$. $O_x \times O_y$ is open in $X \times Y$. O_x and O_y are disjoint, so $(O_x \times O_y) \cap \Delta(X) = \emptyset$. Thus, for an arbitrary point $(x, y) \in (X \times X) \backslash \Delta(X)$, we have evidenced an open set ($O_x \times O_y$) strictly contained within $(X \times X) \backslash \Delta(X)$. Thus $(X \times X) \backslash \Delta(X)$ is open, which tells us that $\Delta(X)$ is closed.

Assume that $\Delta(X)$ is closed. Let $x, y \in X$ such that $x \neq y$. $(x, y) \notin \Delta(X)$ so $(x, y) \in (X \times X) \backslash \Delta(X)$ (which is open), so (x, y) is in N, an open neighborhood of (x, y) that is disjoint from $\Delta(X)$.

N is open in $X \times X$ so $N = \bigcup_\alpha (U_\alpha \times V_\alpha)$ with all U_α and V_α open in X. In each case, $U_\alpha \cap V_\alpha = \emptyset$ for if not then N would not be disjoint from $\Delta(X)$. So, for some α there are open sets U_α and V_α so that $x \in U_\alpha$ and $y \in V_\alpha$; if not, then (x, y) would not be in N. The sets U_α and V_α are open sets separating x and y, so X is Hausdorff.

(b) Check that any affine variety, A, is separated.

A is a closed irreducible subset of some space $\mathbb{A}^n$, so we can view our variety as being in n variables $x_1, \ldots, x_n$. We thus have $\Delta(A) = \{(a, a) | a \in A\} \subset A \times A$. Let our point be $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in A \times A$. $\Delta(A)$ is closed, as it is the zero set of the ideal generated by the polynomials $x_i - y_i$, with $i \in \{1, \ldots, n\}$.

(c) Show that the double-origined $\mathbb{A}^1$ (which we'll call B) is not separated.

Let's adopt the convention that B was obtained by gluing two copies of $\mathbb{A}^1$ together (other than the origin, of course): We'll call these two copies $\mathbb{A}^1_x$ and $\mathbb{A}^1_y$. These two are identified together, other than the origins, which we'll call 0_x and 0_y.

$B \times B = \left(\left(\mathbb{A}^1 \backslash \{0\}\right) \times \left(\mathbb{A}^1 \backslash \{0\}\right)\right) \cup \left\{(0_x, 0_x), (0_x, 0_y), (0_y, 0_x), (0_y, 0_y)\right\}$.

Now examine $\Delta(B)$: We have $\left\{(0_x, 0_x), (0_y, 0_y)\right\} \subset \Delta(B)$, but $(0_x, 0_y) \notin \Delta(B)$ and $(0_y, 0_x) \notin \Delta(B)$. All four origins are in the closure of $\Delta(B)$, so $\Delta(B)$ is not closed.

(d) Verify that $\mathbb{P}^1$ is separated.

We wish to show that $\Delta(\mathbb{P}^1) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is closed. Using the Segre embedding, we have $([x_0 : x_1], [y_0 : y_1]) \mapsto [x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1] = [z_{0,0} : z_{0,1} : z_{1,0} : z_{1,1}]$. $\Delta(\mathbb{P}^1) = \left(\mathbb{P}^1 \times \mathbb{P}^1\right) \cap V\left((z_{0,1} - z_{1,0})\right)$, which is closed, so $\mathbb{P}^1$ is separated.

9. Let $f : C \to C$ be defined as $(a,b) \mapsto \left(-\frac{1}{a}, -\frac{b}{a^2}\right)$. Show $f \circ f$ is the identity map.

$$\begin{aligned} f \circ f\,(a,b) &= f\left(-\frac{1}{a}, -\frac{b}{a^2}\right) \\ &= \left(-\frac{1}{-\frac{1}{a}}, -\frac{-\frac{b}{a^2}}{\left(-\frac{1}{a}\right)^2}\right) \\ &= (a,b) \end{aligned}$$

So $f \circ f$ is the identity map.