1. (a) Give an example of a decent presheaf that is not a sheaf.

   example:

   Let $X = \mathbb{R}$ under the standard topology, and for all open $U \subset \mathbb{R}$, let $\mathcal{F}(U)$ be the bounded continuous functions on U, with restriction being functional restriction (so this is trivially a presheaf). Similarly, the presheaf is trivially decent, as agreement on any open covering of $\mathbb{R}$ implies agreement on every point in $\mathbb{R}$. Let $U_i = \{x \in \mathbb{R} \mid |x| < i\}$. Clearly $\mathbb{R} = \bigcup_j U_j$, and each U_j is open. Note that $i(x) = x$ is in each $\mathcal{F}(X)$, but $i(x)$ is not bounded on $\mathbb{R}$, so $i(x)$ is not in $\mathcal{F}(X)$, so $\mathcal{F}$ is not a sheaf.

   (b) Give an example of a presheaf that is glueable but is indecent.

   example:

   Let $X = \{a, b\}$, and let $A = \{a\}$ and $B = \{b\}$ under the discrete topology. Define $\mathcal{F}(X) = \mathbb{R}\times\mathbb{R}\times\mathbb{R}$, $\mathcal{F}(A) = \mathbb{R}\times 0\times 0$ and $\mathcal{F}(B) = 0\times\mathbb{R}\times 0$ and $\mathcal{F}(\emptyset) = (0, 0, 0)$, with $\mathrm{res}_{X,A}(\cdot) = (\pi_1(\cdot), 0, 0)$ and $\mathrm{res}_{X,B}(\cdot) = (0, \pi_2(\cdot), 0)$ (where π_1 and π_2 are the projections to the first and second space, respectively), and the restriction from a space to itself is the identity map. The restriction functions trivially fulfill the transitivity condition for presheaves.

   We can cover $X = A \cup B$, but $\sigma = (1, 0, 3) \in \mathcal{F}(X) \neq \tau = (1, 0, 4) \in \mathcal{F}(X)$, but $\sigma_a = \tau_a \in \mathcal{F}(A)$ and $\sigma_b = \tau_b \in \mathcal{F}(B)$, so this presheaf is indecent. For glueability, note that there are only four sets possible, each of which is open; any covering of any of these can be built into a section on this open set, though as the pre-sheaf is indecent this construction may not be unique.

3. Let X be the set of all prime numbers $p \in \mathbb{Z}$ under the cofinite topology. Define the presheaf $\mathcal{F}$ for any open set $U \subset X$ as

$$\mathcal{F}(U) = \left\{a \in \mathbb{Q} \mid v_p(a) \geq 0 \text{ for all } p \in U\right\}$$

   using inclusion as the required restriction operation.

   (a) Is $\mathcal{F}$ a sheaf?

   Yes. It is a pre-sheaf under this restriction trivially. To see that this sheaf is decent, simply note that for open $U \subset X$, covered as $U = \bigcup_i U_i$, and for any i, $x_1, x_2 \in \mathcal{F}(U)$, $\mathrm{res}_{U,U_i} x_1 = \mathrm{res}_{U,U_i} x_2 \iff x_1 = x_2$. So, equality in any portion of the cover implies global equality, thus the sheaf is decent.

   Glueability is similarly simple. Let $U \subset X$ be open, covered by open sets $U = \bigcup_i U_i$. Assume we have sections $x_i \in \mathcal{F}(U_i)$ such that $\mathrm{res}_{U_i, U_i \cap U_j} x_i = \mathrm{res}_{U_j, U_i \cap U_j}$. The U_i are an open cover for U, and U is connected (as the only clopen sets in the cofinite topology are X and $\emptyset$), so it must be the case that $x_i = x_j$ for all i, j. Choose an x_i arbitrary and call it x; this x can the thought of as a section for all of U whose restriction to U_i is x_i for all i.

(b) What is $\mathcal{F}(X)$?

In general, if $U \subset X$, $U \neq \emptyset$ is open then $X \backslash U = \{p_1, \ldots, p_k\}$ where p_i are primes.

$$\mathcal{F}(U) = \left\{ \frac{a}{b} \mid a \in \mathbb{Z}, b = p_1^{\ell_1} \cdots p_k^{\ell_k}, \ell_i \text{ are non-negative integers} \right\}$$

We then see that the relationship between U and $\mathcal{F}(U)$ is contravariant: by including a prime in U, we prevent this prime from occurring in the denominator (of the reduced form) of any value in $\mathcal{F}(U)$. Thus $\mathcal{F}(\emptyset) = \mathbb{Q}$, and $\mathcal{F}(X) = \mathbb{Z}$.

(c) What is the stalk of $\mathcal{F}$ at 7?

$$\mathcal{F}_7 = \varinjlim_{U \ni 7} \mathcal{F}(U)$$

Any prime other than 7 has an open set in this direct limit that does not contain that prime. Thus $\mathcal{F}_7 = \mathbb{Q} \backslash \left\{ \frac{a}{7^j} \mid a \in \mathbb{Z}, 7 \not| a, j \in \mathbb{Z}^+ \right\}$.

4. Let X denote a topological space, and let $\mathbb{R}$ be viewed under the Euclidian topology. Which of the following presheaves are sheaves?

a. Continuous real-valued functions on open subsets of $\mathbb{R}$.

This is a sheaf.

b. Bounded and continuous real-valued functions on open subsets of $\mathbb{R}$.

This is not a sheaf (it does not possess the gluing property, as demonstrated in #1).

c. Differentiable real-valued functions on open subsets of $\mathbb{R}$.

This is a sheaf.

d. $\mathcal{F}$ is the presheaf on X defined by $\mathcal{F}(U) = \begin{cases} \mathbb{R} & x \in U \\ 0 & \text{otherwise} \end{cases}$

Assuming that there are non-empty open sets in X that do not contain x, this is not a sheaf.

e. $\mathcal{F}$ is the presheaf on X defined by $\mathcal{F}(U) = \mathbb{R}$ for all $U \neq \emptyset$.

For the sake of completeness, lets assume that this was intended to mean

$$\mathcal{F}(U) = \begin{cases} \mathbb{R} & U \neq \emptyset \\ 0 & U = \emptyset \end{cases}$$

Case $X = \emptyset$: this is an uninteresting sheaf.

Case $X \neq \emptyset$: this is not a sheaf. (e.g., in a two-point space under the discrete topology, there would be no gluing).

5. a. Let X/k denote a variety, and let $\mathcal{O}_X$ denote the sheaf of regular functions on X. Prove that for all $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.

Locally a variety is affine, so $\mathcal{O}_{X,x}$ can be thought of as a localization of a ring of polynomials at a prime ideal, that is $\mathcal{O}_{X,x} \cong k[x_1, \ldots, x_n]_\mathfrak{p}$. Clearly, we have that $k[x_1, \ldots, x_n]_\mathfrak{p}/\mathfrak{p}$ is a field (as all elements outside of $\mathfrak{p}$ are already forced to be units by the localization!), so $\mathfrak{p}$ is a maximal ideal. By the above observation, any element not in $\mathfrak{p}$ is a unit, thus cannot be in a maximal ideal, thus $k[x_1, \ldots, x_n]_\mathfrak{p}$ has a unique maximal ideal, and $k[x_1, \ldots, x_n]_\mathfrak{p}$ is a local ring.

b. Give an example of a topological space X and a sheaf $\mathcal{F}$ on X such that not all stalks $\mathcal{F}_x$ are local rings.

Let $X = \{a\}$, a space with one element, with the only topology possible in this case (all subsets are open).

Let our sheaf be $\mathcal{F}(X) = \mathbb{R}[x]$. $\mathcal{F}_a = \mathcal{F}(X) = \mathbb{R}[x]$, which has an infinite number of maximal ideals (one per irreducible polynomial!).

9. a. Consider the inclusion $f : \{x\} \to X$ of a point into a topological space. Let $\mathcal{F}$ denote a sheaf on the one-point space $\{x\}$. Describe the pushforward sheaf $f_*(\mathcal{F})$ on X.

If $U \subset X$ open, then $f_*(\mathcal{F})(U) = \mathcal{F}\left(f^{-1}(U)\right)$. This reduces to

$$f_*(\mathcal{F})(U) = \begin{cases} \mathcal{F}(\{x\}) & f(x) \in U \\ \mathcal{F}(\emptyset) & \text{otherwise} \end{cases}$$

b. Conversely, consider the constant map $g : X \to \{x\}$ onto a one-point space. Let $\mathcal{G}$ denote a sheaf on X. Describe the pushforward sheaf $g_*(\mathcal{G})$ on $\{x\}$.

If $U \subset \{x\}$ (all subsets are necessarily open), then $g_*(\mathcal{G})(U) = \mathcal{G}\left(g^{-1}(U)\right)$. This reduces to

$$g_*(\mathcal{G})(U) = \begin{cases} \mathcal{G}(X) & U = \{x\} \\ \mathcal{G}(\emptyset) & U = \emptyset \end{cases}$$