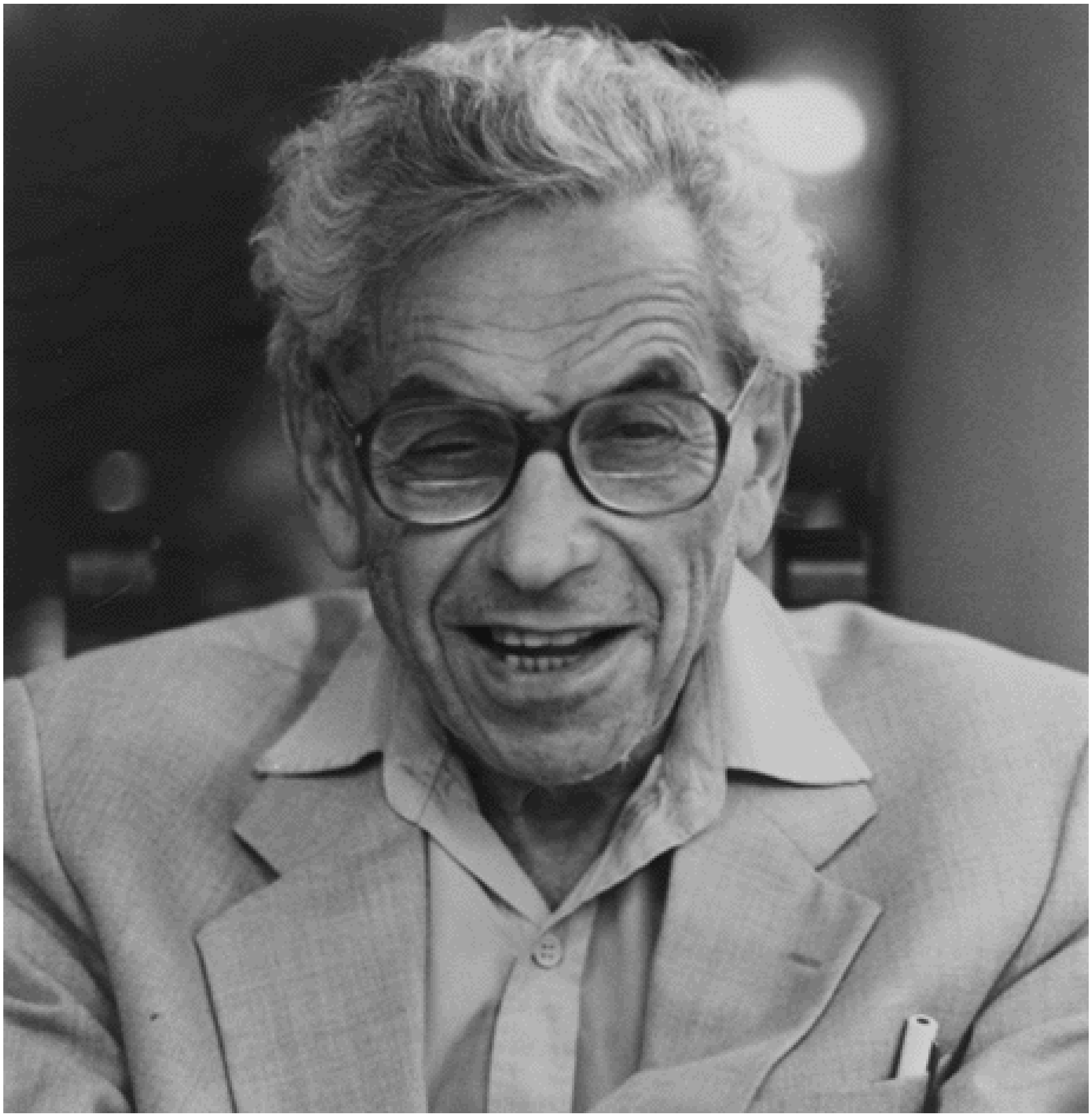

Paul Erdős
Mathematical Genius, Human
(In That Order)

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MATH 341-01
4 June 2004

"A Mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent that theirs, it is because the are made with ideas... The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colours of the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics." --G.H. Hardy

"Why are numbers beautiful? It's like asking why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is." -- Paul Erdős

"One of the first people I met in Princeton was Paul Erdős. He was 26 years old at the time, and had his Ph.D. for several years, and had been bouncing from one postdoctoral fellowship to another... Though I was slightly younger, I considered myself wiser in the ways of the world, and I lectured Erdős "This fellowship business is all well and good, but it can't go on for much longer -- jobs are hard to get -- you had better get on the ball and start looking for a real honest job." ... Forty years after my sermon, Erdős hasn't found it necessary to look for an "honest" job yet." -- Paul Halmos

Introduction

Paul Erdős (said "Air-daish") was a brilliant and prolific mathematician, who was central to the advancement of several major branches of mathematics. His contributions to the mathematical community have served to advance the borders of human knowledge, both as a brilliant explorer in his own right, and also in an additional way: Erdős acted as a centrally-situated director of research. He encouraged others to proceed in fruitful areas, both by direct collaboration and through encouragement.

Erdős was productive throughout his entire professional life, from 1932 at the age of 18 when he published his first paper, until 2003, almost 7 years after his death as straggling papers were published posthumously. Because of this unusual degree of consistency in the level of mathematical output, Erdős is often cited as a counterexample to Hardy's famous quote:

> "No mathematician should ever allow himself to forget that mathematics, more than any other art or science, is a young man's game... I do not know an instance of a major mathematical advance initiated by a man past fifty." -- G.H. Hardy

Erdős attained this remarkable output through many devices, principal among them rampant collaboration. He tirelessly collaborated with other mathematicians, often in sessions involving as many a dozen mathematicians simultaneously working on different problems, splitting his time much like a grandmaster at a chess camp, moving from mathematician to mathematician, effortlessly moving between problems, leaving his collaborators time to think (and rest!) between sessions of the intense thought and concentration that Erdős both cultivated in himself, and demanded from others.

> "That first day we did mathematics until one in the morning. I went upstairs to bed, and he stayed downstairs in the guest room. At 4:30 A.M. I heard pots banging in the kitchen. He kept banging them. It was his way of telling me to get up. I stumbled downstairs about six. What were the first words out of his mouth? Not 'Good morning' or 'How'd you sleep?', but 'Let n be an integer. Suppose k is...'" -- Mike Plummer

Upon reflection, one may wonder how any individual manages to keep this sort of frantic pace throughout their life. In Erdős's instance, there are a couple of readily apparent answers: First, Erdős's life was wholly devoted to mathematics. He did not have a job, a regular domicile, or more possessions than he could carry with him in his two (half empty!) suitcases. He traveled from university to university, from mathematician to mathematician, working until his collaborator was exhausted, and then moving on. He did not cultivate human contact outside of his mathematical interactions, with the exception of his Mother, whom he loved dearly. He didn't have to cook, clean or accomplish any of the other range of daily tasks that might sap the time and energy that he could devote to mathematics. Instead, he had a cadre of people who looked after him, and saw to it that he had food, shelter and, incidentally, a visa for his next destination.

The other, somewhat darker, answer to this question is "drugs". Paul Erdős was a habitual amphetamine user the last 20 years of his life, and a heavy consumer of other more mainstream stimulants his entire life.

But beyond his skill at collaboration, his apparently unending energy, and his voracious appetite for discovery, Erdős was, incidentally, also a mathematical genius of his own right. There are a legion of stories from independent sources that go something like this one:

> "In 1976, we were having coffee in the mathematics lounge at Texas A&M. There was a problem on the blackboard in functional analysis, a field Erdős knew nothing about. I happened to know that two analysts had just come up with a thirty-page solution to the problem and were very proud of it. Erdős looked up at the board and said 'What's that? Is it a problem?' I said yes, and he went up to the board and squinted at the tersely written statement. He asked a few questions about what the symbols represented, and then he effortlessly wrote down a two-line solution. If that's not magic, what is?" -- George Purdy

Given this consuming (and incidentally, very effective) focus on mathematics, one might reasonably ask how Erdős interacted with others who either had to, from time to time, surface in order to deal with trifling details like "food", "shelter" or other notable entries on Maslow's Hierarchy, or (even worse!) non-mathematicians. The answer is principally "he didn't". Erdős had command of the child's skill of warping reality around himself. Others had to deal with food, but Erdős did not. Others had to converse in understandable ways with people without doctorates in mathematics, but not Erdős. Erdős separated himself from the so called "trivial beings" (i.e. non-mathematicians), not only using the impenetrable jargon of mathematics, but also through his use of his own contrived language, where women were "bosses", men were "slaves", children were "epsilons", people "arrived" rather than being born, people "died" when they stopped doing mathematics, and people "left" when they died. Erdős didn't lecture, he "preached", and when he was ready to do mathematics, "his brain was open". Finally, as he treated God as somewhat of an opponent, it was "The Supreme Fascist" (or "SF"), rather than God.

Paul Erdős with his Mother

Human

Paul Erdős was born 26 March 1913 in Budapest Hungary. Paul's mother, Anna, and father, Lajos were both school teachers. His family was ethnically Jewish, though non-practicing. Erdős's older sisters died of scarlet fever while his mother was in the hospital giving birth to him. As a consequence of this loss, his mother was very protective of Paul his entire life. She refused to send him to public schools, instead retaining a private tutor for Paul's entire primary school education. Paul attended the public high school during alternating years, owing to his mother's indecision.

World War I broke out when Paul was a year old. Paul's father served in the Austria-Hungarian army, and was captured by the Russian army early in the war. His father spent 6 years in a Siberian prison, where he taught himself English by picking up the language from available English novels. When his imprisonment was over and he returned to Hungary, he proceeded to teach Paul English. Without prolonged exposure to any native

English speakers, Paul learned grammatically proper English, but his accent was very distinctive and challenging to understand.

Paul was shown to be a mathematical prodigy very early in his life; he could multiply 3 digit numbers and had independently developed the idea of "negative numbers" by the age of three. He gained local acclaim, and tutored several other mathematical prodigies in the community.

The political climate in Hungary was chaotic after World War I. After a brief time as a democratic republic, the government briefly changed to a violent communist regime headed by Béla Kun, who was ethnically Jewish. This government was crushed by an even more violent fascist regime headed by Miklós Horthy, who identified the Jewish community as supporters of the previous Kun regime, and so actively oppressed the group. People identified as Jewish were not allowed to attend the university system, and so Paul could not attend the university until 1930, when he won a national examination, and was thus exempt from the anti-Jewish measures.

This brush with fascism had a lasting impact on Paul, both politically and linguistically. From this point on, anything was inconvenient or oppressive was "fascist". To illustrate: At one point, a colleague showed Erdős a new litter of kittens. Paul picked up a kitten, but the kitten would have none of it, and scratched him. Paul carefully put the kitten back down, clucked and proclaimed "fascist cat!". Paul's colleague, intrigued, asked Paul how the cat could possibly be fascist. Paul responded, "If you were a mouse, you would know!".

Paul attended the University Pazmany Peter in Budapest from 1930 to 1934, when he left with his Ph.D. in Mathematics. At this point, he was interested in fleeing the repressive fascist government, so he left Hungary (and his family) in 1934, and went to England for a postdoctoral fellowship at Manchester. This move seemed to stoke Paul's wanderlust, and he started his mathematical wanderings. During this time Paul also worked in Cambridge, London, Bristol and others. While at Cambridge, Erdős met Stanislaw Ulam and G.H. Hardy at Cambridge.

By 1938 he could no longer safely return to Hungary because of Hitler's control of Austria, so Erdős decided to go further afield. He first went to Princeton for a yearlong fellowship at the Institute for Advanced Study.

While at Princeton, Erdős met Paul Halmos, another Hungarian émigré. Halmos's remembrance of Erdős was as follows:

> "It would happen that someone would ask him a question in a field that he knew nothing about; he then demanded that the basic words be defined for him, and if it turned out that his set-theoretic "counting" techniques were at all pertinent, he proceeded to find the answer. A typical example is the dimension of the set of rational points in Hilbert space, a problem that Hurewicz raised. Erdős was a little vague about what Hilbert space was, and he had no idea what "dimension" meant. Harry Wallman gave him the definitions -- and before long Erdős produced the

> solution. (The answer is 1.) The paper came out in the Annals in 1940; it is a technically important contribution to a subject that a few months earlier Erdős knew nothing about." -- Paul Halmos

While at Princeton, Hungary allied itself with Germany, and Paul lost the ability to communicate with his family.

After his year term, Princeton refused to renew his fellowship for an additional full year; Princeton found him "Uncouth and unconventional", and only offered to extend his fellowship for 6 months. Paul left Princeton, and started wandering, university-to-university, mathematician-to-mathematician, and conference-to-conference.

In 1941, Erdős had a minor run-in with the US Government when Erdős and two other mathematicians absentmindedly wandered onto a secret radar facility after overlooking a "No Trespassing" sign. A facility guard asked them to leave, which they did without incident. The guard thought their behavior odd, so he reported the incident, and the FBI questioned the mathematicians about the incident later. This resulted in Erdős having a FBI file, which eventually became a source of trouble.

In 1943, Ulam tried to convince Erdős to work for the US government at Los Alamos on the Nuclear weapons project that was in process there. Erdős was interested in helping in the process (and in helping to destroy the fascist governments that had assumed control throughout eastern Europe), but he wasn't suitably careful in his security interviews, indicating that he was interested in returning to Hungary (after all, his mother and family was there). In addition, Erdős's contrariness helped disqualify him; he deliberately nettled the military security personal by loudly making inquiries in public like "How is work going on the A-bomb?"

In 1945 Soviet troops liberated Budapest. Erdős received word that most of his extended family had been killed in Auschwitz and that Erdős's father had died of a heart attack in 1942. Erdős visited Hungary, and spent a great deal of time going between England and the US.

Beginning in 1954, Erdős started having problems with the American authorities when the US refused to issue an entrance visa for Paul. As a result, Erdős spent much of the 1950s in Israel. A pattern emerged where the US State Department would refuse to issue an entrance visa, then the mathematical community would circulate a petition requesting Erdős's admittance for some particular conference. Sometimes the State department would relent for a limited time, but for the most part they did not. Erdős joked that "the US State Department is adamant on two points: non-admission of Red China tot the U.N. and non-admission of Paul Erdős to the U.S."

In 1955, Erdős's travel restrictions were eased somewhat when he was issued a unique Hungarian passport that allowed him to freely enter and leave Hungary. This was the only passport of this kind issued.

Erdős employed a series of helpers to keep his travel-laden schedule running well. Principal among these helpers was Ronald Graham, who managed his estate. Erdős didn't concern himself with trivialities like "money", so Graham had to: "I signed his name on checks and deposited them. I did this so long I doubt the bank would have cashed a check if he had endorsed it himself."

In 1964 his mother (who was then 84) started traveling with Erdős and taking care of him. During this time, Paul and his mother were inseparable. They continued to travel with each other until 1971, when Erdős's mother died of a bleeding ulcer after she was misdiagnosed. Immediately after his mother's death, Paul started taking large amount of antidepressants and then later amphetamines. Erdős attempted to regain some degree of equilibrium in his life by burying himself in mathematics, consistently spending 19 hours a day in mathematical pursuits.

His mother's death continued to trouble Paul for years; Paul would bring his mother up at odd times:

> "I was walking across a courtyard to breakfast at a conference and Erdős, who had just had breakfast, was walking in the opposite direction. When our paths crossed, I offered my customary greeting, 'Good morning, Paul. How are you today?' He stopped dead in his tracks. Out of respect and deference, I stopped too. We just stood there silently.... Finally, after much reflection, he said: 'Herbert, today I am very sad.' And I said, 'I am sorry to hear that. Why are you sad, Paul?' He said, 'I am sad because I miss my mother. She is dead you know.' I said, 'I know that, Paul. I know her death was very sad for you and for many of us, too. But wasn't that about five years ago?' He said, 'Yes, it was. But I miss her very much.'" -- Herb Wilf

Erdős's friends worried about his drug use, and in 1979 Graham bet Erdős $500 that he couldn't stop taking amphetamines for a month. Erdős accepted, and went cold turkey for a complete month. Erdős's comment at the end of the month was "You've showed me I'm not an addict. But I didn't get any work done. I'd get up in the morning and stare at a blank piece of paper. I'd have no ideas, just like an ordinary person. You've set mathematics back a month." He then immediately started taking amphetamines again.

As Erdős didn't "suffer" anything that would detract from his effectiveness as a mathematician, he didn't have much in the way of social graces. Mathematics was his first priority, and if you wanted to collaborate with him, then you had to, at least temporarily adopt that same priority. He generally lacked any regard for others' schedules, as having to respect others' schedule would detract from his mathematical pursuits:

> "When the name of a colleague in California came up at breakfast in New Jersey, Erdős remembered a mathematical result he wanted to share with him. He headed toward the phone and started to dial. His host interrupted him, pointing out that it was 5:00 A.M. on the west coast. "Good," Erdős said, "That means he'll be home." -- Paul Hoffman

For that matter, though it was important to know some contact information for his collaborators, he didn't necessarily need to know much else about them, including their first name.

> "He called me all the time. 'Is professor Winkler there?'... He knew every mathematician's phone number, but I don't think he knew anyone's first name. I doubt if he would have recognized my first name even though I worked with him for twenty years. The only person he called by his first name was Tom Trotter, whom he called Bill." -- Peter Winkler

In addition, as many of Erdős's collaborations were handled via mail, and because he dealt with so many people he would sometimes forget what they actually looked like. On one occasion, Erdős met a mathematician and asked him where he was from. "Vancouver," the mathematician replied. "Oh, then you must know my good friend Elliot Mendelson", Erdős said. The reply was "I AM your good friend Elliot Mendelson."

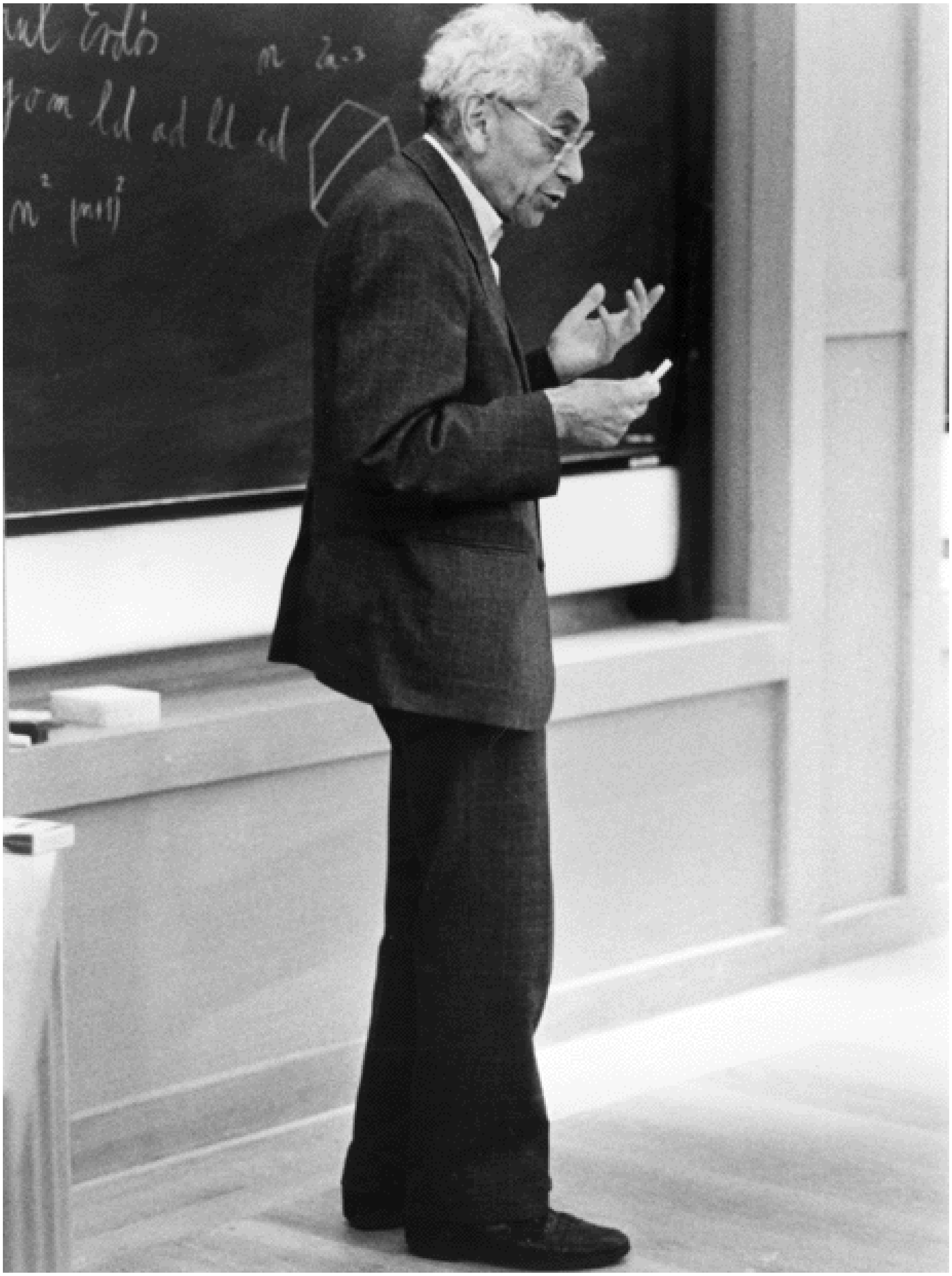
As Erdős never had to care for his own day-to-day existence, he never really picked up skills that most people take for granted.

> "Once I spent a few days with Paul. When I entered the kitchen in the evening, I was met with a horrible sight. The floor was covered by pools of blood-like red liquid. The trail led to the refrigerator. I opened the door, and to my great surprise saw a carton of tomato juice on its side with a gaping hole. Paul must have felt thirsty and, after some reflection, decided to get the juice out of the carton by stabbing it with a big knife." -- János Pach

In fact, Erdős lacked the ability to prepare any but the most basic foods for himself. He couldn't cook for himself, or even boil water for tea. He once mentioned, "I can make excellent cold cereal, and I could probably boil an egg, but I've never tried." He was 21 when he first buttered his own bread. "I remember clearly, I had just gone to England to study. It was teatime, and bread was served. I was too embarrassed to admit that I had never buttered it. I tried. It wasn't so hard."

Erdős's travel schedule forced Paul to adopt a minimalist approach to personal property. Erdős was quoted as saying "Some French socialist said that private property was theft. I say that private property is a nuisance."

Money given to him through stipends and lecture fees were given away to relatives, colleagues, students and strangers. He rarely kept very much for himself, as he had very modest needs. In 1984 he won the Wolf Prize, which included an award of $50,000. He set up a scholarship in Israel in the name of his parents, and kept only $750 of the prize money.



Paul Erdős "Preaching"

Mathematician

"A Mathematician is a machine for turning coffee into theorems." -- Paul Erdős

World, Meet Paul Erdős

Paul Erdős's first paper was a new proof of Bertrand's conjecture, which was published during Erdős's Freshman year at college in an article titled "Beweis eines Satzes von Tschebyschef" in the journal Acta Litt. Sci Szeged 5 (1932), 195-198; Zbl. 4,101. Bertrand's Conjecture was first proven by Tschebyschef, in 1850. Unfortunately, Tschebyschef 's proof was very long and involved, and was not considered particularly elegant. Erdős's proof stunned the international mathematical community with its brevity and elegance. It gave world its first taste of a proof style that was to become indelibly associated with Paul Erdős.

Theorem (*Bertrand's Conjecture*):

If $n \geq 1$, there is at least one prime p such that $n < p \leq 2n$.

Proof: First, note that Bertrand's conjecture is satisfied by each of the numbers up to 512. The primes:

2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631

can be used to satisfy Bertrand's conjecture for all numbers up to 512. This implies that if Bertrand's Conjecture fails, it must fail for a number greater than 512.

Proof by contradiction: Assume that for some $n > 2^9 = 512$, there is no prime between n and 2n.

First, we need a Lemma:

Lemma 1: $n! = \prod_p p^{j(n,p)}$, where $j(n,p) = \sum_{m \geq 1}^{\infty} \left[\frac{n}{p^m} \right]$

Proof: The numbers 1, 2, 3, …, n include [n / p] multiples of p, and $[n / p^2]$ multiples of p^2 and so on. This is true for each prime up to n. □

Note: Though j(n, p) appears to be an infinite series, its terms are integer values that go to 0, as soon p^m is greater than n. Thus this could equivalently be expressed as the busier finite sum to $\left[\frac{\log n}{\log p} \right]$, which more obviously converges (but whose notation is a bit busier).

Example: Using this lemma, $6! = 2^{j(6,\,2)}3^{j(6,\,3)}5^{j(6,5)}$

$j(6,2) = \left[\frac{6}{2^1} \right] + \left[\frac{6}{2^2} \right] = 3 + 1$, $j(6,3) = \left[\frac{6}{3^1} \right] = 2$, $j(6,5) = \left[\frac{6}{5^1} \right] = 1$

$6! = 2^4 3^2 5^1$

Note: $6! = (1)(2)(3)(4)(5)(6) = (1)(2)(3)((2)(2))(5)((2)(3)) = 2^4 3^2 5^1$

Let $N = \frac{(2n)!}{(n!)^2}$. By lemma 1, this can be written as $N = \frac{\prod_{p \le 2n} p^{j(2n,p)}}{\left(\prod_{p \le n} p^{j(n,p)}\right)^2} = \frac{\prod_{p \le 2n} p^{j(2n,p)}}{\prod_{p \le n} p^{2j(n,p)}}$

Because the j(n, p) becomes 0 when p is greater than n, it is harmless to combine these expressions to: $N = \prod_{p \le 2n} p^{j(2n,p)-2j(n,p)} = \prod_{p \le 2n} p^{k_p}$, where $k_p = \sum_{m=1}^{\infty} \left(\left[\frac{2n}{p^m} \right] - 2\left[\frac{n}{p^m} \right] \right)$. Note that each term of the summation that composes k_p is either 0 or 1, depending on whether $[2n / p^m]$ is even or odd, respectively. Because there are at most $\left[\frac{\log 2n}{\log p} \right]$ terms in k_p, and each term is at most 1, we know that

$$\text{[1]} \ k_p \le \left[\frac{\log 2n}{\log p} \right].$$

Now, let p be a prime factor of N. This implies that $k_p \ge 1$. (If $k_p = 0$, p would not be a prime factor of N). By our hypothesis, $p \le n$. ($p \le 2n$, by the definition of N. If $n < p \le 2n$, then this prime would satisfy Bertrand's conjecture for n, whereas we have assumed that there is no such prime for this value of n).

It must be the case that $p \le \frac{2}{3}n$. If this were not the case, then $\frac{2}{3}n < p \le n$, and thus $2n < 3p \le 3n$. Thus $2p \le 2n < 3p$. Additionally, $\frac{4}{9}n^2 < p^2 \le n^2$. Because n>512, $2n < \frac{4}{9}n^2 < p^2$.

This implies that $k_p = \left[\frac{2n}{p} \right] - 2\left[\frac{n}{p} \right] = 2 - 2 = 0$, and thus p would not be a factor of N. This same reasoning can be applied to every prime factor of N, implying that every prime factor of N is also less than $\frac{2}{3}n$.

Let $\vartheta(x) = \sum_{p \le x} \log p = \log \prod_{p \le x} p$.

Lemma 2: $(\forall n \in \mathbb{Z}^+) \vartheta(n) < 2n \log 2$

Proof:

First, let's establish a fact that will be necessary in proving this lemma:

Let $M = \binom{2m+1}{m} = \frac{(2m+1)!}{m!(m+1)!}$. M is an integer which occurs twice in the binomial expansion of $(1+1)^{2m+1}$. As a result, $2M < 2^{2m+1}$ or $M<2^{2m}$.

If m+1 < p ≤ 2m + 1, p divides the numerator of M, but not the denominator, and thus p divides M. This is true for all primes that satisfy this inequality (and all these primes are relatively prime to each other), so

$$\left(\prod_{m+1<p\le 2m+1} p\right) | M \text{ and } \vartheta(2m+1)-\vartheta(m+1)=\sum_{m+1<p\le 2m+1}\log p \le \log M < 2m\log 2.$$

This inequality is used in the proof of the lemma, which is a proof through induction; Lemma 2 is true for n=1 and n=2.

Assume that this Lemma is true for all n less than some integer, $n_0>2$.

If n_0 is even, we have:

$$\vartheta(n_0)=\vartheta(n_0-1)<2(n_0-1)\log 2 = 2n_0\log 2 - 2\log 2 < 2n_0\log 2$$

(remember that we are only concerned with the prime numbers less than or equal to n_0; if n_0 is even and greater than 2, it can't be prime, so subtracting one does not change value of the function)

If n_0 is odd, let $n_0=2m+1$, we have:

$$\vartheta(n_0)=\vartheta(2m+1)=\vartheta(2m+1)-\vartheta(m+1)+\vartheta(m+1)$$
$$< 2m\log 2 + 2(m+1)\log 2 = 2(2m+1)\log 2 = 2n_0\log 2.$$

(m+1 < 2m+1=n_0 so the application of the inductive step here is valid)

As a consequence, $\vartheta(n_0)<2n_0\log 2$, so this lemma holds for all positive integers. □

By lemma 2, $\sum_{p|n}\log p \le \sum_{p\le \frac{2}{3}n}\log p = \vartheta\left(\frac{2}{3}n\right) < \frac{4}{3}n\log 2$.

If $k_p\geq 2$, by equation *[1]*, we have

$$2\log p \le k_p\log p \le \log(2n), \text{ which implies } p\le\sqrt{2n}.$$

This implies that there are at most $\sqrt{2n}$ values of p where $k_p\geq 2$. Hence, $\sum_{k_p\ge 2} k_p\log p \le \sqrt{2n}\log(2n)$, and also

$$\log N \le \sum_{k_p=1}\log p + \sum_{k_p\ge 2}k_p\log p \le \sum_{k_p=1}\log p + \sqrt{2n}\log(2n) \le \frac{4}{3}n\log 2 + \sqrt{2n}\log(2n).$$

If we look at N in terms of the binomial theorem, we note that N is that largest term for the expansion $(1+1)^{2n} = 2^{2n}$. As a consequence of this, we see that

$$2^{2n} = 2 + \binom{2n}{1} + \binom{2n}{2} + \cdots + \binom{2n}{2n-1} \le 2nN.$$

Combining the two previous inequalities, we get

$$2n\log 2 \le \log(2n) + \log N \le \frac{4}{3}n\log 2 + (1+\sqrt{2n})\log(2n), \text{ which reduces to}$$

$$\textit{[2]}\quad 2n\log 2 \le 3(1+\sqrt{2n})\log(2n).$$

For the sake of brevity (and for the added prestige of using additional Greek letters), we write $\zeta = \dfrac{\log(n/512)}{10\log 2} > 0$ (as n>512).

Note that $2n = 2^{10(1+\zeta)}$. Rewriting *[2]* using this, we see that $2^{10(1+\zeta)} \leq 30(2^{5+5\zeta}+1)(1+\zeta)$, and so…

$$2^{5\zeta} \leq 30\cdot 2^{-5}(1+2^{-5-5\zeta})(1+\zeta) < (1+2^{-5})(1+2^{-5})(1+\zeta) < 1+\zeta$$

But wait!

$$2^{5\zeta} = \exp(5\zeta\log 2) > 1+5\zeta\log 2 > 1+\zeta$$

Which is a contradiction!

So n cannot be equal to or smaller than 512, or greater than 512 given our assumptions, therefore our assumptions are contradictory, and therefore there does exist a prime between any number and double that number, and Bertrand's Conjecture holds. □

Corollary: $p_{r+1} < 2p_r$

Let n equal p_r. Bertrand's Conjecture gives us $p_r < p_{r+1} \leq 2p_r$, but p_{r+1} is an odd prime (and thus can not be a multiple of 2), so $p_{r+1} < 2p_r$. □

Other Contributions

Erdős had a very strong sense of mathematical aesthetics that strongly affected all of his work. He didn't just want a solution; he wanted an elegant solution that tied together apparently dissimilar concepts. This feeling of mathematical aesthetics took on almost religious significance to Erdős.

> "I'm not qualified to say whether or not God exists. I kind of doubt He does. Nevertheless, I'm always saying that the SF has this transfinite Book that contains the best proofs of all mathematical theorems, proofs that are elegant and perfect… You don't have to believe in God, but you should believe in the Book." -- Paul Erdős

A collection of these "perfect proofs" was assembled by Martin Aigner and Gunter Ziegler (with help from Erdős) into a book called Proofs from the Book. This book was first intended to be presented to Erdős on his 85th birthday, but unfortunately Erdős died before the book was complete. Proofs from the Book was first published in 1998, and is currently in its third edition. It is dedicated to the memory of Paul Erdős.

This strong sense of aesthetics didn't prevent Erdős from publishing prolifically. Erdős is credited as an author for 1521 academic papers (as of Feb 2004), which makes him the most prolific mathematician ever (even more so than Euler). Erdős has collaborated with 509 authors, nearly twice as many as the next most well connected mathematician. Erdős collaborated so much that the most accepted measure of connectedness for a mathematician to the mathematical community (particularly the number theory and discrete mathematics community) is based upon the journal separation between the subject and Erdős, the mathematician's "Erdős number". Erdős had a Erdős number of 0. The lucky 509 mathematicians who had co-written a paper with Erdős have an Erdős number of 1. Mathematicians that have co-written an article with one of this group (but not Erdős, himself) have an Erdős number of 2, and so on. If there is no article path from a person to Erdős, then that person has an infinite Erdős number. The distribution of Erdős numbers is as follows:

Erdős Number	Number of People
0	1
1	509
2	6984
3	26422
4	62136
5	66157
6	32280
7	10431
8	3214
9	953
10	262
11	94
12	23
13	4
14	7
15	1
16	0

The median Erdős number is 5, the mean is 4.69 and the standard deviation is 1.27

There are a number of related systems, ranging from serious (Erdős number of the second kind, which only counts papers with two authors) to whimsical (for example, a person's Bacon-Erdős number can be obtained by determining that person's movie separation from Kevin Bacon, and then adding that person's mathematical journal separation from Paul Erdős)

A low Erdős number is considered prestigious in some circles. There was even an instance where a moderately low Erdős number was sold on e-bay. (i.e. The holder of an Erdős number offered to collaborate with the winner of the auction)

In addition to being directly productive, Erdős directed the mathematical community in several ways: He actively sought out young prodigies, and collaborated with them. He was generally able to guide them into becoming successful members of the math community, but there were several notable instances where his protégés became distracted from mathematics and "died".

Erdős also setup prizes for a series of mathematical problems that he couldn't find solutions for. He set the level of the prize based on his estimate of the difficulty of the problems. The prize size varied between $1 and $1,000. A number of these problems are still outstanding and the he prize money is still being provided by Erdős's estate and (in some cases) other interested parties.

Lastly, Erdős would happily share his opinion of the difficulty of problems with proto-mathematicians who approached him, sometimes discouraging them from pursuing their current problem if he judged it impossibly difficult to solve. "I went up [to Erdős] and told him the problem I was working on in combinatorial set theory. He responded, without a moment's hesitation, 'I think you need to find another problem… 21 years later, that problem is still open". -- Michael Jacobson

The experience of collaborating with Erdős was generally very positive. John Selfridge said this of producing the proof that the product of consecutive integers is never a square, cube or any higher power with Erdős: "I was there working with him, improving a few of the things he did, putting my two cents' worth in. But I told him, 'This is really your solution. I only helped you.' He didn't exactly like that. He like the idea that this was a joint paper, so it was."

There were a few instances where people were effectively forced into collaborating with Erdős with poor results. The most notable of these incidents involved the "elementary" proof of the Prime number theorem. In this instance, a colleague of Selberg described some recent advancement to Erdős, who very quickly jumped ahead of Selberg in one particular portion of the proof, ultimately resulting in a proof of The Prime Number Theorem. Selberg, who was not happy with Erdős's involvement, quickly developed an alternate version of the proof that did not rely on Erdős's methods, and subsequently won a Fields medal for his work in this area.

Another such incident involved a publication editor challenging Erdős to improve upon a proof by Bellman and Shapario that was provided to Erdős for the purpose of refereeing a journal submission. In this instance, Erdős improved upon the submitted proof substantially, but the original authors did not want to provide Erdős co-author status. This particular issue was resolved by not publishing any of the work, either the original proof, or Erdős's much improved proof.

Coda

Erdős felt that his mathematical abilities were declining near the end of his life because he started having difficulty following the arguments that he used in his papers 40 years previous He started to use Graham to check his work; Graham reported that he usually still didn't make mistakes.

In March 1996, Erdős lost consciousness while giving a lecture. After regaining consciousness a short time later, he asked audience not to leave insisting, "I have two more problems to tell them".

In June 1996, Erdős attended a conference and was listening to Gerhart Ringel deliver a talk at a conference. At the end of the presentation, Erdős started to ask a question, but lost consciousness before he could complete the question. He was rushed into surgery and had a pacemaker implanted. He managed to attend the closing banquet for the conference, with his heart surgeons in tow. At the closing banquet, he introduced his heart surgeons, took a bow, and proceeded to finish his question.

Paul Erdős died 20 September 1996 after having two heart attacks. His funeral was held in Budapest. Erdős's memorial had over 500 people present. Paul was cremated and his remains were buried next to his mother's remains.

> "Végre nem butulok tovább" (Finally I am becoming stupider no more) -- The epitaph Paul Erdős wrote for himself.

> "Paul Erdős was one of those very special geniuses, the kind who comes along only once in a very long while yet he chose, quite consciously I am sure, to share mathematics with mere mortals -- like me. And for this, I will always be grateful to him. I will miss the times he prowled my hallways at 4:00 A.M., and came to my bed to ask whether my "brain is open." I will miss the problems and conjectures and the stimulating conversations about anything and everything. But most of all, I will just miss Paul, the human. I loved him dearly." -- Tom Trotter

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