# Maximum of the Sum of Additive Characters Over the Images of Polynomials

In $\mathbb{F}_q$ (with $q = p^m$) additive characters are all of the form $\psi_\gamma : \mathbb{F}_q \to \mathbb{C}$.

$$\psi_\gamma(\alpha) = e^{\frac{2\pi i}{p}\mathrm{Tr}(\gamma\alpha)}$$

for some $\gamma \in \mathbb{F}_q^*$ and with the trace being the absolute trace

$$\mathrm{Tr}(\alpha) = \mathrm{Tr}_{\mathbb{F}_p}^{\mathbb{F}_q}(\alpha) = \sum_{j=0}^{m-1} \alpha^{p^j}$$

Take $f \in \mathbb{F}_q[x]$, a polynomial of degree $d$ with $p \not| d$. Define

$$V_f = \{f(\beta) \mid \beta \in \mathbb{F}_q\}$$

Define

$$S_f = \sum_{\alpha \in V_f} \psi_\gamma(\alpha)$$

*Conjecture*: for all polynomial functions of degree $d$, there is a real number $c_d$ so that for any $\gamma \in \mathbb{F}_q^*$, we have

$$\left|S_f\right| \leq c_d \sqrt{q}$$

In principal, we are looking at polynomials of the form

$$p(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$$

One does not have to look at all polynomials of this form in order to find all the values of $\left|S_f\right|$.

$p(x-\lambda)$ has the same image as $p(x)$ for all values of $\lambda \in \mathbb{F}_q$ (as we are simply rearranging the order that we plug values into the function).

Examine the $x^{d-1}$ term of (the expanded) $p(x-\lambda)$; there are only two contributors to this term: the $a_d(x-\lambda)^d$ term contributes $-a_d d\lambda x^{d-1}$ and the $a_{d-1}(x-\lambda)^{d-1}$ term contributes $a_{d-1}x^{d-1}$ (by the binomial theorem). The $x^{d-1}$ term is then $(a_{d-1} - da_d\lambda)x^{d-1}$. The choice of $\lambda = \frac{a_{d-1}}{da_d}$ then results in a polynomial with no $x^{d-1}$ term (we know this is possible, as $a_d \neq 0$ as $p$ was assumed to be of degree $d$).

Thus we can, without loss of generality, assume that our initial polynomial had no $x^{d-1}$ term.

Now note that if we have a generic polynomial of this form

$$p(x) = a_d x^d + a_{d-2}x^{d-2} + \cdots + a_1 x + a_0$$

Consider

$$\begin{aligned}
\left|S_f\right| &= \left|\sum_{\beta\in\mathbb{F}_q} \psi_\gamma\left(p\left(\beta\right)\right)\right| \\
&= \left|\sum_{\beta\in\mathbb{F}_q} \psi_\gamma\left(a_d\beta^d + a_{d-2}\beta^{d-2} + \cdots + a_1\beta + a_0\right)\right| \\
&= \left|\sum_{\beta\in\mathbb{F}_q} \psi_\gamma\left(a_d\beta^d + a_{d-2}\beta^{d-2} + \cdots + a_1\beta\right)\psi_\gamma\left(a_0\right)\right| \\
&= \left|\psi_\gamma\left(a_0\right)\sum_{\beta\in\mathbb{F}_q} \psi_\gamma\left(a_d\beta^d + a_{d-2}\beta^{d-2} + \cdots + a_1\beta\right)\right| \\
&= \left|\psi_\gamma\left(a_0\right)\right|\left|\sum_{\beta\in\mathbb{F}_q} \psi_\gamma\left(a_d\beta^d + a_{d-2}\beta^{d-2} + \cdots + a_1\beta\right)\right| \\
&= \left|\sum_{\beta\in\mathbb{F}_q} \psi_\gamma\left(a_d\beta^d + a_{d-2}\beta^{d-2} + \cdots + a_1\beta\right)\right| \\
&= \left|\sum_{\beta\in\mathbb{F}_q} \psi_\gamma\left(a_d\left(\beta^d + \frac{a_{d-2}}{a_d}\beta^{d-2} + \cdots + \frac{a_1}{a_d}\beta\right)\right)\right| \\
&= \left|\sum_{\beta\in\mathbb{F}_q} \psi_{\gamma a_d}\left(\beta^d + \frac{a_{d-2}}{a_d}\beta^{d-2} + \cdots + \frac{a_1}{a_d}\beta\right)\right|
\end{aligned}$$

As we are already looking across all possible additive characters (and thus are taking the maximum across all possible $\gamma$), the $a_d$ term in $\gamma a_d$ doesn't affect anything (it simply permutes the ordering of the characters...), so we have thus concluded that all possible $\left|S_f\right|$ values are encountered by simply examining polynomials of the form

$$p(x) = x^d + a_{d-2}x^{d-2} + \cdots + a_1 x \text{ with } a_i \in \mathbb{F}_q$$

To determine what reasonable bounds on $c_d$ are, we examine:

$$\Phi_d = \max_{\substack{\gamma \in \mathbb{F}_q^* \\ f \in \mathbb{F}_q[x] \\ \deg f = d}} \frac{\left|S_f\right|}{\sqrt{q}}$$

In Table 1, $\Phi_d$ is estimated by randomly choosing 1000 polynomials of degree $d$ (up to $d = q$) from $\mathbb{F}_q[x]$ of the form above and then calculating the maximum value observed for $\Phi_d$.

In Table 2, $\Phi_d$ is calculated directly by calculating $\left|S_f\right|$ for every polynomial $f$ of that degree, and for every $\gamma$. The unshaded values present in Table 2 are the values from Table 1, and are present for context only.

Items marked 0 are less than $10^{-9}$.

Items market * have been excluded because $p|d$.

**Table 1a:** Estimates for $\Phi_d$ (Random Generic Polynomials Method)

| | Polynomial Degree ($d$) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $q$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 0 | * | | | | | | |
| 3 | 0 | 0.57735 | * | | | | | |
| 4 | 0 | * | 1.00000 | * | | | | |
| 5 | 0 | 0.72361 | 0.72361 | 0.72361 | * | | | |
| 7 | 0 | 0.53452 | 0.84928 | 0.84928 | 0.84928 | 0.84928 | * | |
| 8 | 0 | * | 1.06066 | * | 1.41421 | * | 1.41421 | * |
| 9 | 0 | 0.66667 | * | 1.00000 | 1.00000 | * | 1.00000 | 1.00000 |
| 11 | 0 | 0.52223 | 0.97349 | 0.82199 | 1.05931 | 1.05931 | 1.05931 | 1.05931 |
| 13 | 0 | 0.63867 | 0.95378 | 0.95378 | 1.15048 | 1.15048 | 1.15048 | 1.15048 |
| 16 | 0 | * | 0.75000 | * | 2.00000 | * | 1.75000 | * |
| 17 | 0 | 0.62127 | 1.02700 | 1.02700 | 1.18406 | 1.26954 | 1.26954 | 1.26954 |
| 19 | 0 | 0.51299 | 1.04701 | 1.13259 | 1.20765 | 1.27810 | 1.35117 | 1.31381 |
| 23 | 0 | 0.51075 | 1.03430 | 1.14760 | 1.34400 | 1.44374 | 1.35889 | 1.24339 |
| 25 | 0 | 0.60000 | 0.84721 | 1.04721 | * | 1.37082 | 1.37082 | 1.37082 |
| 27 | 0 | 0.50918 | * | 1.52753 | 1.38778 | * | 1.34715 | 1.34715 |
| 29 | 0 | 0.59285 | 1.07385 | 1.02658 | 1.42638 | 1.43852 | 1.31657 | 1.35800 |
| 31 | 0 | 0.50800 | 1.09672 | 1.22594 | 1.52480 | 1.57718 | 1.36677 | 1.52829 |
| 32 | 0 | * | 0.88388 | * | 1.94454 | * | 1.94454 | * |
| 37 | 0 | 0.58220 | 0.98716 | 1.24637 | 1.52525 | 1.64751 | 1.56248 | 1.44715 |
| 41 | 0 | 0.57809 | 1.08314 | 1.22734 | 1.50484 | 1.50229 | 1.58844 | 1.41488 |
| 43 | 0 | 0.50578 | 1.19704 | 1.30236 | 1.52521 | 1.35157 | 1.49797 | 1.40729 |
| 47 | 0 | 0.50529 | 1.01460 | 1.29680 | 1.54358 | 1.42041 | 1.47739 | 1.49281 |
| 49 | 0 | 0.57143 | 0.93470 | 1.23978 | 1.54356 | 1.47731 | * | 1.47783 |
| 53 | 0 | 0.56868 | 0.98112 | 1.55491 | 1.54066 | 1.66984 | 1.77354 | 1.56766 |
| 59 | 0 | 0.50422 | 1.18476 | 1.23615 | 1.50171 | 1.45484 | 1.66832 | 1.73034 |
| 61 | 0 | 0.56402 | 1.01759 | 1.27676 | 1.48530 | 1.40667 | 1.62327 | 1.61360 |
| 64 | 0 | * | 1.12500 | * | 2.00000 | * | 2.25000 | * |
| 67 | 0 | 0.50372 | 1.08765 | 1.28408 | 1.60184 | 1.49948 | 1.49986 | 1.40041 |
| 71 | 0 | 0.50351 | 1.09729 | 1.39732 | 1.65384 | 1.62178 | 1.46927 | 1.65889 |
| 73 | 0 | 0.55852 | 1.25796 | 1.37843 | 1.58441 | 1.61481 | 1.43867 | 1.46433 |
| 79 | 0 | 0.50316 | 1.02226 | 1.40891 | 1.62967 | 1.56628 | 1.65248 | 1.64895 |
| 81 | 0 | 0.55556 | * | 1.41857 | 1.65924 | * | 1.56742 | 1.56742 |
| 83 | 0 | 0.50300 | 1.10991 | 1.37432 | 1.72373 | 1.44996 | 1.56494 | 1.61410 |
| 89 | 0 | 0.55300 | 1.19288 | 1.29922 | 1.51001 | 1.47747 | 1.53146 | 1.50130 |
| 97 | 0 | 0.55077 | 1.15420 | 1.40274 | 1.52626 | 1.46688 | 1.61483 | 1.65230 |

**Table 1b:** Estimates for $\Phi_d$ (Random Generic Polynomials Method)

| | Polynomial Degree ($d$) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $q$ | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 2 | | | | | | | | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | | | |
| 7 | | | | | | | | |
| 8 | | | | | | | | |
| 9 | * | | | | | | | |
| 11 | 1.05931 | 1.05931 | * | | | | | |
| 13 | 1.15048 | 1.15048 | 1.15048 | 1.15048 | * | | | |
| 16 | 1.75000 | * | 1.75000 | * | 1.75000 | * | 1.75000 | * |
| 17 | 1.26954 | 1.31429 | 1.26954 | 1.26954 | 1.31429 | 1.26954 | 1.26954 | 1.26954 |
| 19 | 1.35117 | 1.31381 | 1.27810 | 1.27643 | 1.27643 | 1.27810 | 1.35117 | 1.38906 |
| 23 | 1.33294 | 1.29715 | 1.31067 | 1.38706 | 1.49929 | 1.38952 | 1.44374 | 1.31634 |
| 25 | 1.42799 | * | 1.33780 | 1.42263 | 1.32276 | 1.30394 | * | 1.32276 |
| 27 | * | 1.38778 | 1.34715 | * | 1.38778 | 1.34715 | * | 1.45297 |
| 29 | 1.35125 | 1.33174 | 1.44649 | 1.44109 | 1.46003 | 1.40073 | 1.36849 | 1.46096 |
| 31 | 1.41091 | 1.39870 | 1.33020 | 1.33779 | 1.54627 | 1.40707 | 1.40626 | 1.40477 |
| 32 | 1.94454 | * | 1.94454 | * | 2.12132 | * | 2.12132 | * |
| 37 | 1.54435 | 1.35317 | 1.42731 | 1.35638 | 1.41973 | 1.46984 | 1.55660 | 1.38932 |
| 41 | 1.62118 | 1.46018 | 1.47270 | 1.50872 | 1.57470 | 1.37584 | 1.49152 | 1.57665 |
| 43 | 1.62093 | 1.53455 | 1.48371 | 1.46242 | 1.46950 | 1.58422 | 1.38042 | 1.68610 |
| 47 | 1.43187 | 1.52571 | 1.58231 | 1.43541 | 1.44183 | 1.52809 | 1.55981 | 1.48418 |
| 49 | 1.58342 | 1.54682 | 1.47700 | 1.44198 | 1.40199 | * | 1.50520 | 1.52109 |
| 53 | 1.51378 | 1.62999 | 1.51634 | 1.49741 | 1.47345 | 1.54307 | 1.53005 | 1.47544 |
| 59 | 1.48861 | 1.53492 | 1.51775 | 1.41585 | 1.62089 | 1.42924 | 1.61098 | 1.52077 |
| 61 | 1.45774 | 1.70483 | 1.67307 | 1.61652 | 1.46420 | 1.47430 | 1.42433 | 1.43009 |
| 64 | 2.00000 | * | 2.00000 | * | 2.12500 | * | 2.00000 | * |
| 67 | 1.45141 | 1.53333 | 1.54263 | 1.59874 | 1.51811 | 1.59312 | 1.50674 | 1.52011 |
| 71 | 1.67909 | 1.54702 | 1.72202 | 1.64874 | 1.46727 | 1.61690 | 1.62799 | 1.53782 |
| 73 | 1.54161 | 1.58043 | 1.48376 | 1.51601 | 1.65409 | 1.57450 | 1.60998 | 1.49381 |
| 79 | 1.50879 | 1.59252 | 1.51211 | 1.59598 | 1.61152 | 1.44245 | 1.62079 | 1.48744 |
| 81 | * | 1.50308 | 1.50308 | * | 1.68142 | 1.61398 | * | 1.61398 |
| 83 | 1.51231 | 1.68314 | 1.59052 | 1.49117 | 1.76893 | 1.45831 | 1.53770 | 1.66690 |
| 89 | 1.48742 | 1.80501 | 1.54485 | 1.54781 | 1.58076 | 1.68566 | 1.54807 | 1.57066 |
| 97 | 1.47133 | 1.54993 | 1.55815 | 1.63828 | 1.51302 | 1.65451 | 1.54286 | 1.69522 |

**Table 1c:** Estimates for $\Phi_d$ (Random Generic Polynomials Method)

| | Polynomial Degree ($d$) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $q$ | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| 2 | | | | | | | | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | | | |
| 7 | | | | | | | | |
| 8 | | | | | | | | |
| 9 | | | | | | | | |
| 11 | | | | | | | | |
| 13 | | | | | | | | |
| 16 | | | | | | | | |
| 17 | * | | | | | | | |
| 19 | 1.35117 | 1.27643 | * | | | | | |
| 23 | 1.33793 | 1.34400 | 1.33793 | 1.33358 | 1.31067 | 1.37232 | * | |
| 25 | 1.44520 | 1.37082 | 1.42799 | * | 1.33780 | 1.39632 | 1.33780 | 1.39632 |
| 27 | 1.52753 | * | 1.38778 | 1.45297 | * | 1.45297 | 1.45297 | * |
| 29 | 1.44835 | 1.55829 | 1.42164 | 1.37768 | 1.54519 | 1.40861 | 1.48208 | 1.37768 |
| 31 | 1.52443 | 1.46877 | 1.46182 | 1.33388 | 1.34640 | 1.51624 | 1.38052 | 1.39910 |
| 32 | 1.76777 | * | 1.94454 | * | 1.94454 | * | 1.94454 | * |
| 37 | 1.43720 | 1.39882 | 1.48055 | 1.40865 | 1.35613 | 1.44353 | 1.54549 | 1.48472 |
| 41 | 1.37389 | 1.49618 | 1.47403 | 1.54622 | 1.44387 | 1.40286 | 1.50011 | 1.49818 |
| 43 | 1.36190 | 1.50986 | 1.45349 | 1.52722 | 1.55568 | 1.47214 | 1.53322 | 1.48095 |
| 47 | 1.44789 | 1.55547 | 1.58066 | 1.49283 | 1.45872 | 1.52084 | 1.52697 | 1.49611 |
| 49 | 1.50761 | 1.73525 | 1.54598 | 1.45316 | * | 1.46140 | 1.55679 | 1.63014 |
| 53 | 1.41959 | 1.56474 | 1.39321 | 1.44852 | 1.53415 | 1.45274 | 1.38172 | 1.58972 |
| 59 | 1.40933 | 1.45788 | 1.37811 | 1.50323 | 1.49667 | 1.44767 | 1.50586 | 1.54294 |
| 61 | 1.42835 | 1.58968 | 1.54380 | 1.51365 | 1.58282 | 1.47075 | 1.48999 | 1.55564 |
| 64 | 2.25000 | * | 2.12500 | * | 2.25000 | * | 2.00000 | * |
| 67 | 1.58744 | 1.50856 | 1.48770 | 1.61059 | 1.42428 | 1.59153 | 1.60685 | 1.50774 |
| 71 | 1.63306 | 1.85298 | 1.60632 | 1.40485 | 1.54725 | 1.56471 | 1.47370 | 1.45671 |
| 73 | 1.54880 | 1.51056 | 1.53260 | 1.50727 | 1.68054 | 1.66545 | 1.44813 | 1.61621 |
| 79 | 1.50450 | 1.65694 | 1.52883 | 1.53308 | 1.52550 | 1.46018 | 1.65006 | 1.56032 |
| 81 | 1.55556 | * | 1.49485 | 1.68142 | * | 1.54360 | 1.61398 | * |
| 83 | 1.78463 | 1.56388 | 1.57968 | 1.60125 | 1.49495 | 1.53318 | 1.65172 | 1.42656 |
| 89 | 1.55068 | 1.66741 | 1.70923 | 1.58543 | 1.57932 | 1.57210 | 1.58046 | 1.63142 |
| 97 | 1.56037 | 1.53282 | 1.51193 | 1.57577 | 1.46869 | 1.68064 | 1.55835 | 1.72851 |

**Table 1d:** Estimates for $\Phi_d$ (Random Generic Polynomials Method)

| | Polynomial Degree ($d$) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $q$ | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 |
| 2 | | | | | | | | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | | | |
| 7 | | | | | | | | |
| 8 | | | | | | | | |
| 9 | | | | | | | | |
| 11 | | | | | | | | |
| 13 | | | | | | | | |
| 16 | | | | | | | | |
| 17 | | | | | | | | |
| 19 | | | | | | | | |
| 23 | | | | | | | | |
| 25 | * | | | | | | | |
| 27 | 1.53960 | 1.34715 | * | | | | | |
| 29 | 1.37768 | 1.35695 | 1.39666 | 1.61517 | * | | | |
| 31 | 1.42693 | 1.44179 | 1.49374 | 1.47144 | 1.32965 | 1.38088 | * | |
| 32 | 2.29810 | * | 1.76777 | * | 1.94454 | * | 1.94454 | * |
| 37 | 1.51486 | 1.42797 | 1.40763 | 1.37045 | 1.63981 | 1.47914 | 1.55157 | 1.51862 |
| 41 | 1.51862 | 1.43834 | 1.44189 | 1.46422 | 1.48092 | 1.44395 | 1.45678 | 1.72341 |
| 43 | 1.61708 | 1.64842 | 1.41887 | 1.41350 | 1.46880 | 1.60807 | 1.64622 | 1.41842 |
| 47 | 1.72185 | 1.42522 | 1.45151 | 1.49440 | 1.65213 | 1.55140 | 1.59036 | 1.37699 |
| 49 | 1.48371 | 1.46093 | 1.42861 | * | 1.46843 | 1.62636 | 1.65462 | 1.42996 |
| 53 | 1.54930 | 1.46887 | 1.39833 | 1.42970 | 1.47393 | 1.51981 | 1.72883 | 1.54675 |
| 59 | 1.46063 | 1.55318 | 1.43170 | 1.46304 | 1.45331 | 1.57923 | 1.55750 | 1.52425 |
| 61 | 1.44631 | 1.53710 | 1.53442 | 1.41538 | 1.51525 | 1.42250 | 1.68349 | 1.44011 |
| 64 | 2.00000 | * | 2.25000 | * | 2.00000 | * | 2.12500 | * |
| 67 | 1.64265 | 1.47007 | 1.45620 | 1.69040 | 1.52521 | 1.42950 | 1.58479 | 1.55817 |
| 71 | 1.49094 | 1.57196 | 1.54591 | 1.54253 | 1.59794 | 1.56588 | 1.47625 | 1.55715 |
| 73 | 1.67815 | 1.52472 | 1.61980 | 1.45744 | 1.41510 | 1.44875 | 1.46015 | 1.48765 |
| 79 | 1.53192 | 1.51375 | 1.67826 | 1.46032 | 1.50953 | 1.54849 | 1.54488 | 1.50125 |
| 81 | 1.52753 | 1.55556 | * | 1.65924 | 1.67774 | * | 1.78816 | 1.73205 |
| 83 | 1.50493 | 1.48252 | 1.46155 | 1.49846 | 1.53297 | 1.54310 | 1.66943 | 1.61359 |
| 89 | 1.60026 | 1.62076 | 1.57532 | 1.51737 | 1.47972 | 1.68020 | 1.52155 | 1.74356 |
| 97 | 1.53520 | 1.63251 | 1.56962 | 1.52219 | 1.84426 | 1.61767 | 1.63560 | 1.53683 |

**Table 1e:** Estimates for $\Phi_d$ (Random Generic Polynomials Method)

| | Polynomial Degree ($d$) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $q$ | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 2 | | | | | | | | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | | | |
| 7 | | | | | | | | |
| 8 | | | | | | | | |
| 9 | | | | | | | | |
| 11 | | | | | | | | |
| 13 | | | | | | | | |
| 16 | | | | | | | | |
| 17 | | | | | | | | |
| 19 | | | | | | | | |
| 23 | | | | | | | | |
| 25 | | | | | | | | |
| 27 | | | | | | | | |
| 29 | | | | | | | | |
| 31 | | | | | | | | |
| 32 | | | | | | | | |
| 37 | 1.42420 | 1.41653 | 1.44523 | 1.49021 | * | | | |
| 41 | 1.51019 | 1.50224 | 1.38610 | 1.50891 | 1.48370 | 1.57527 | 1.51600 | 1.63014 |
| 43 | 1.53201 | 1.39668 | 1.37981 | 1.46549 | 1.56913 | 1.55511 | 1.51373 | 1.42848 |
| 47 | 1.56475 | 1.52553 | 1.43891 | 1.49509 | 1.59545 | 1.46185 | 1.42414 | 1.66604 |
| 49 | 1.59910 | 1.44407 | * | 1.45532 | 1.59056 | 1.49695 | 1.45273 | 1.44777 |
| 53 | 1.46912 | 1.58910 | 1.52231 | 1.63587 | 1.54176 | 1.55947 | 1.50036 | 1.41556 |
| 59 | 1.45629 | 1.63213 | 1.63995 | 1.49164 | 1.57883 | 1.51766 | 1.55959 | 1.49333 |
| 61 | 1.53592 | 1.52356 | 1.48852 | 1.56066 | 1.49539 | 1.50549 | 1.58833 | 1.46516 |
| 64 | 2.12500 | * | 2.12500 | * | 2.00000 | * | 2.12500 | * |
| 67 | 1.56987 | 1.67925 | 1.44329 | 1.53172 | 1.55772 | 1.64780 | 1.61721 | 1.48640 |
| 71 | 1.58278 | 1.61474 | 1.63907 | 1.52025 | 1.53327 | 1.40940 | 1.75331 | 1.52470 |
| 73 | 1.57860 | 1.49293 | 1.69657 | 1.50235 | 1.62455 | 1.45706 | 1.39939 | 1.51419 |
| 79 | 1.71430 | 1.62353 | 1.49412 | 1.67741 | 1.62778 | 1.71710 | 1.72487 | 1.58271 |
| 81 | * | 1.52753 | 1.57527 | * | 1.53960 | 1.55556 | * | 1.57527 |
| 83 | 1.75752 | 1.48521 | 1.59399 | 1.77521 | 1.54289 | 1.66259 | 1.46696 | 1.58811 |
| 89 | 1.63382 | 1.62324 | 1.46041 | 1.61967 | 1.55115 | 1.59574 | 1.47852 | 1.46194 |
| 97 | 1.59471 | 1.70475 | 1.58867 | 1.44627 | 1.71882 | 1.49087 | 1.64175 | 1.57217 |

**Table 1f:** Estimates for $\Phi_d$ (Random Generic Polynomials Method)

| | Polynomial Degree ($d$) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $q$ | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 |
| 2 | | | | | | | | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | | | |
| 7 | | | | | | | | |
| 8 | | | | | | | | |
| 9 | | | | | | | | |
| 11 | | | | | | | | |
| 13 | | | | | | | | |
| 16 | | | | | | | | |
| 17 | | | | | | | | |
| 19 | | | | | | | | |
| 23 | | | | | | | | |
| 25 | | | | | | | | |
| 27 | | | | | | | | |
| 29 | | | | | | | | |
| 31 | | | | | | | | |
| 32 | | | | | | | | |
| 37 | | | | | | | | |
| 41 | * | | | | | | | |
| 43 | 1.40240 | 1.46303 | * | | | | | |
| 47 | 1.52674 | 1.44628 | 1.57420 | 1.53338 | 1.41104 | 1.54641 | * | |
| 49 | 1.64357 | * | 1.57633 | 1.45402 | 1.43805 | 1.52260 | 1.44472 | 1.61002 |
| 53 | 1.57065 | 1.75984 | 1.43103 | 1.48614 | 1.64308 | 1.59873 | 1.61372 | 1.43957 |
| 59 | 1.51218 | 1.42421 | 1.53819 | 1.49538 | 1.45099 | 1.47054 | 1.47385 | 1.45498 |
| 61 | 1.62274 | 1.52670 | 1.67637 | 1.56711 | 1.38634 | 1.64313 | 1.57623 | 1.64891 |
| 64 | 2.12500 | * | 2.12500 | * | 2.00000 | * | 2.00000 | * |
| 67 | 1.49350 | 1.53006 | 1.49153 | 1.51215 | 1.61459 | 1.48564 | 1.48913 | 1.59778 |
| 71 | 1.47836 | 1.58079 | 1.54719 | 1.50138 | 1.46732 | 1.50544 | 1.68596 | 1.47508 |
| 73 | 1.69807 | 1.46910 | 1.50349 | 1.62314 | 1.47431 | 1.42808 | 1.59728 | 1.54753 |
| 79 | 1.55474 | 1.44912 | 1.41887 | 1.51228 | 1.54114 | 1.63536 | 1.72290 | 1.54874 |
| 81 | 1.54360 | * | 1.52753 | 1.46986 | * | 1.52753 | 1.65924 | * |
| 83 | 1.47848 | 1.61633 | 1.59562 | 1.59597 | 1.54992 | 1.52447 | 1.45062 | 1.49928 |
| 89 | 1.57617 | 1.63456 | 1.70340 | 1.67675 | 1.61801 | 1.48831 | 1.51393 | 1.62530 |
| 97 | 1.59419 | 1.52187 | 1.53490 | 1.52159 | 1.52699 | 1.59250 | 1.69896 | 1.58543 |

**Table 1g:** Estimates for $\Phi_d$ (Random Generic Polynomials Method)

| | Polynomial Degree ($d$) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $q$ | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 |
| 2 | | | | | | | | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | | | |
| 7 | | | | | | | | |
| 8 | | | | | | | | |
| 9 | | | | | | | | |
| 11 | | | | | | | | |
| 13 | | | | | | | | |
| 16 | | | | | | | | |
| 17 | | | | | | | | |
| 19 | | | | | | | | |
| 23 | | | | | | | | |
| 25 | | | | | | | | |
| 27 | | | | | | | | |
| 29 | | | | | | | | |
| 31 | | | | | | | | |
| 32 | | | | | | | | |
| 37 | | | | | | | | |
| 41 | | | | | | | | |
| 43 | | | | | | | | |
| 47 | | | | | | | | |
| 49 | * | | | | | | | |
| 53 | 1.75812 | 1.41853 | 1.54826 | 1.45873 | * | | | |
| 59 | 1.51843 | 1.56346 | 1.53375 | 1.40338 | 1.50439 | 1.44423 | 1.49556 | 1.56841 |
| 61 | 1.47631 | 1.53688 | 1.65840 | 1.46056 | 1.51249 | 1.44346 | 1.51600 | 1.54998 |
| 64 | 2.00000 | * | 2.12500 | * | 2.12500 | * | 1.75000 | * |
| 67 | 1.45364 | 1.61324 | 1.58429 | 1.42772 | 1.40730 | 1.64946 | 1.76135 | 1.60834 |
| 71 | 1.46370 | 1.53282 | 1.50885 | 1.54432 | 1.48440 | 1.62591 | 1.53003 | 1.69061 |
| 73 | 1.60742 | 1.51356 | 1.44458 | 1.52048 | 1.39142 | 1.47147 | 1.56393 | 1.50371 |
| 79 | 1.54594 | 1.58004 | 1.75888 | 1.50162 | 1.41857 | 1.44116 | 1.62446 | 1.54354 |
| 81 | 1.56742 | 1.64429 | * | 1.78816 | 1.63677 | * | 1.49485 | 1.54360 |
| 83 | 1.50368 | 1.49717 | 1.60252 | 1.63612 | 1.59553 | 1.48743 | 1.59896 | 1.50927 |
| 89 | 1.49582 | 1.61441 | 1.49972 | 1.62829 | 1.60212 | 1.70545 | 1.63433 | 1.56328 |
| 97 | 1.52276 | 1.60668 | 1.56938 | 1.51217 | 1.69451 | 1.55840 | 1.62149 | 1.55880 |

**Table 1h:** Estimates for $\Phi_d$ (Random Generic Polynomials Method)

| | Polynomial Degree ($d$) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $q$ | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 |
| 2 | | | | | | | | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | | | |
| 7 | | | | | | | | |
| 8 | | | | | | | | |
| 9 | | | | | | | | |
| 11 | | | | | | | | |
| 13 | | | | | | | | |
| 16 | | | | | | | | |
| 17 | | | | | | | | |
| 19 | | | | | | | | |
| 23 | | | | | | | | |
| 25 | | | | | | | | |
| 27 | | | | | | | | |
| 29 | | | | | | | | |
| 31 | | | | | | | | |
| 32 | | | | | | | | |
| 37 | | | | | | | | |
| 41 | | | | | | | | |
| 43 | | | | | | | | |
| 47 | | | | | | | | |
| 49 | | | | | | | | |
| 53 | | | | | | | | |
| 59 | 1.47286 | 1.48982 | * | | | | | |
| 61 | 1.50244 | 1.67305 | 1.53758 | 1.56015 | * | | | |
| 64 | 2.25000 | * | 2.12500 | * | 2.37500 | * | 2.12500 | * |
| 67 | 1.49815 | 1.60777 | 1.53018 | 1.69661 | 1.56412 | 1.55585 | 1.45165 | 1.56080 |
| 71 | 1.68852 | 1.53747 | 1.43330 | 1.52792 | 1.48449 | 1.71163 | 1.48499 | 1.53355 |
| 73 | 1.57989 | 1.49741 | 1.55636 | 1.68907 | 1.54579 | 1.55475 | 1.67184 | 1.53526 |
| 79 | 1.67444 | 1.53339 | 1.54895 | 1.72009 | 1.45540 | 1.50340 | 1.46011 | 1.56185 |
| 81 | * | 1.45721 | 1.54360 | * | 1.52753 | 1.50308 | * | 1.55556 |
| 83 | 1.48538 | 1.65649 | 1.55224 | 1.53431 | 1.59556 | 1.77004 | 1.70554 | 1.70825 |
| 89 | 1.41202 | 1.54167 | 1.60387 | 1.63289 | 1.54632 | 1.64535 | 1.64014 | 1.67776 |
| 97 | 1.61124 | 1.62225 | 1.51175 | 1.63450 | 1.51725 | 1.76693 | 1.61938 | 1.51764 |

**Table 1i:** Estimates for $\Phi_d$ (Random Generic Polynomials Method)

| | Polynomial Degree ($d$) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $q$ | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 |
| 2 | | | | | | | | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | | | |
| 7 | | | | | | | | |
| 8 | | | | | | | | |
| 9 | | | | | | | | |
| 11 | | | | | | | | |
| 13 | | | | | | | | |
| 16 | | | | | | | | |
| 17 | | | | | | | | |
| 19 | | | | | | | | |
| 23 | | | | | | | | |
| 25 | | | | | | | | |
| 27 | | | | | | | | |
| 29 | | | | | | | | |
| 31 | | | | | | | | |
| 32 | | | | | | | | |
| 37 | | | | | | | | |
| 41 | | | | | | | | |
| 43 | | | | | | | | |
| 47 | | | | | | | | |
| 49 | | | | | | | | |
| 53 | | | | | | | | |
| 59 | | | | | | | | |
| 61 | | | | | | | | |
| 64 | | | | | | | | |
| 67 | 1.57671 | 1.50166 | * | | | | | |
| 71 | 1.53202 | 1.53443 | 1.47870 | 1.47667 | 1.46885 | 1.52176 | * | |
| 73 | 1.61726 | 1.58626 | 1.54098 | 1.58484 | 1.42079 | 1.50308 | 1.44779 | 1.55077 |
| 79 | 1.49362 | 1.46169 | 1.57174 | 1.48207 | 1.46635 | 1.46226 | 1.53081 | 1.52579 |
| 81 | 1.56742 | * | 1.64429 | 1.52753 | * | 1.52753 | 1.55556 | * |
| 83 | 1.54225 | 1.54581 | 1.63068 | 1.52563 | 1.54306 | 1.43517 | 1.49239 | 1.42508 |
| 89 | 1.68424 | 1.54246 | 1.52955 | 1.50603 | 1.68456 | 1.45603 | 1.51559 | 1.63310 |
| 97 | 1.66451 | 1.63909 | 1.52793 | 1.63734 | 1.49615 | 1.50927 | 1.73107 | 1.56485 |

**Table 1j:** Estimates for $\Phi_d$ (Random Generic Polynomials Method)

| | Polynomial Degree ($d$) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $q$ | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 2 | | | | | | | | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | | | |
| 7 | | | | | | | | |
| 8 | | | | | | | | |
| 9 | | | | | | | | |
| 11 | | | | | | | | |
| 13 | | | | | | | | |
| 16 | | | | | | | | |
| 17 | | | | | | | | |
| 19 | | | | | | | | |
| 23 | | | | | | | | |
| 25 | | | | | | | | |
| 27 | | | | | | | | |
| 29 | | | | | | | | |
| 31 | | | | | | | | |
| 32 | | | | | | | | |
| 37 | | | | | | | | |
| 41 | | | | | | | | |
| 43 | | | | | | | | |
| 47 | | | | | | | | |
| 49 | | | | | | | | |
| 53 | | | | | | | | |
| 59 | | | | | | | | |
| 61 | | | | | | | | |
| 64 | | | | | | | | |
| 67 | | | | | | | | |
| 71 | | | | | | | | |
| 73 | * | | | | | | | |
| 79 | 1.55544 | 1.46187 | 1.48837 | 1.59823 | 1.50103 | 1.50974 | * | |
| 81 | 1.45297 | 1.63677 | * | 1.63677 | 1.65924 | * | 1.68142 | 1.57527 |
| 83 | 1.62608 | 1.54012 | 1.57953 | 1.56431 | 1.44547 | 1.53882 | 1.55401 | 1.59538 |
| 89 | 1.60407 | 1.58438 | 1.60977 | 1.58777 | 1.62772 | 1.65258 | 1.54351 | 1.62179 |
| 97 | 1.50385 | 1.57105 | 1.55328 | 1.62375 | 1.54366 | 1.55856 | 1.55702 | 1.50762 |

**Table 1k:** Estimates for $\Phi_d$ (Random Generic Polynomials Method)

| | Polynomial Degree ($d$) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $q$ | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 |
| 2 | | | | | | | | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | | | |
| 7 | | | | | | | | |
| 8 | | | | | | | | |
| 9 | | | | | | | | |
| 11 | | | | | | | | |
| 13 | | | | | | | | |
| 16 | | | | | | | | |
| 17 | | | | | | | | |
| 19 | | | | | | | | |
| 23 | | | | | | | | |
| 25 | | | | | | | | |
| 27 | | | | | | | | |
| 29 | | | | | | | | |
| 31 | | | | | | | | |
| 32 | | | | | | | | |
| 37 | | | | | | | | |
| 41 | | | | | | | | |
| 43 | | | | | | | | |
| 47 | | | | | | | | |
| 49 | | | | | | | | |
| 53 | | | | | | | | |
| 59 | | | | | | | | |
| 61 | | | | | | | | |
| 64 | | | | | | | | |
| 67 | | | | | | | | |
| 71 | | | | | | | | |
| 73 | | | | | | | | |
| 79 | | | | | | | | |
| 81 | * | | | | | | | |
| 83 | 1.53175 | 1.49564 | * | | | | | |
| 89 | 1.53851 | 1.58183 | 1.50869 | 1.45390 | 1.57353 | 1.59330 | 1.65806 | 1.52758 |
| 97 | 1.58499 | 1.54645 | 1.55753 | 1.57581 | 1.55842 | 1.66089 | 1.61411 | 1.57416 |

**Table 1l:** Estimates for $\Phi_d$ (Random Generic Polynomials Method)

| | Polynomial Degree ($d$) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $q$ | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 |
| 2 | | | | | | | | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | | | |
| 7 | | | | | | | | |
| 8 | | | | | | | | |
| 9 | | | | | | | | |
| 11 | | | | | | | | |
| 13 | | | | | | | | |
| 16 | | | | | | | | |
| 17 | | | | | | | | |
| 19 | | | | | | | | |
| 23 | | | | | | | | |
| 25 | | | | | | | | |
| 27 | | | | | | | | |
| 29 | | | | | | | | |
| 31 | | | | | | | | |
| 32 | | | | | | | | |
| 37 | | | | | | | | |
| 41 | | | | | | | | |
| 43 | | | | | | | | |
| 47 | | | | | | | | |
| 49 | | | | | | | | |
| 53 | | | | | | | | |
| 59 | | | | | | | | |
| 61 | | | | | | | | |
| 64 | | | | | | | | |
| 67 | | | | | | | | |
| 71 | | | | | | | | |
| 73 | | | | | | | | |
| 79 | | | | | | | | |
| 81 | | | | | | | | |
| 83 | | | | | | | | |
| 89 | * | | | | | | | |
| 97 | 1.54502 | 1.60155 | 1.46050 | 1.53564 | 1.69184 | 1.73105 | 1.53554 | 1.55209 |

**Table 2:** Actual $\Phi_d$ values (in color; blue agrees with Table 1 estimates, red does not)

| | Polynomial Degree ($d$) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $q$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 0 | * | | | | | | |
| 3 | 0 | 0.57735 | * | | | | | |
| 4 | 0 | * | 1.00000 | * | | | | |
| 5 | 0 | 0.72361 | 0.72361 | 0.72361 | * | | | |
| 7 | 0 | 0.53452 | 0.84928 | 0.84928 | 0.84928 | 0.84928 | * | |
| 8 | 0 | * | 1.06066 | * | 1.41421 | * | 1.41421 | * |
| 9 | 0 | 0.66667 | * | 1.00000 | 1.00000 | * | 1.00000 | 1.00000 |
| 11 | 0 | 0.52223 | 0.97349 | 0.82199 | 1.05931 | 1.05931 | 1.05931 | 1.05931 |
| 13 | 0 | 0.63867 | 0.95378 | 0.95378 | 1.15048 | 1.15048 | 1.15048 | 1.15048 |
| 16 | 0 | * | 0.75000 | * | 2.00000 | * | 1.75000 | * |
| 17 | 0 | 0.62127 | 1.02700 | 1.02700 | 1.31429 | 1.31429 | 1.31429 | 1.26954 |
| 19 | 0 | 0.51299 | 1.04701 | 1.13259 | 1.20765 | 1.38906 | 1.38906 | 1.31381 |
| 23 | 0 | 0.51075 | 1.03430 | 1.14760 | 1.49929 | 1.44374 | 1.52775 | 1.24339 |
| 25 | 0 | 0.60000 | 0.84721 | 1.04721 | * | 1.37082 | 1.37082 | 1.37082 |
| 27 | 0 | 0.50918 | * | 1.52753 | 1.38778 | * | 1.34715 | 1.34715 |
| 29 | 0 | 0.59285 | 1.07385 | 1.02658 | 1.42638 | 1.43852 | 1.31657 | 1.35800 |
| 31 | 0 | 0.50800 | 1.09672 | 1.22594 | 1.52480 | 1.57718 | 1.36677 | 1.52829 |
| 32 | 0 | * | 0.88388 | * | 1.94454 | * | 1.94454 | * |
| 37 | 0 | 0.58220 | 0.98716 | 1.24637 | 1.52525 | 1.64751 | 1.56248 | 1.44715 |
| 41 | 0 | 0.57809 | 1.08314 | 1.22734 | 1.50484 | 1.50229 | 1.58844 | 1.41488 |
| 43 | 0 | 0.50578 | 1.19704 | 1.30236 | 1.52521 | 1.35157 | 1.49797 | 1.40729 |
| 47 | 0 | 0.50529 | 1.01460 | 1.29680 | 1.54358 | 1.42041 | 1.47739 | 1.49281 |
| 49 | 0 | 0.57143 | 0.93470 | 1.23978 | 1.54356 | 1.47731 | * | 1.47783 |
| 53 | 0 | 0.56868 | 0.98112 | 1.55491 | 1.54066 | 1.66984 | 1.77354 | 1.56766 |
| 59 | 0 | 0.50422 | 1.18476 | 1.23615 | 1.50171 | 1.45484 | 1.66832 | 1.73034 |
| 61 | 0 | 0.56402 | 1.01759 | 1.27676 | 1.48530 | 1.40667 | 1.62327 | 1.61360 |
| 64 | 0 | * | 1.12500 | * | 2.00000 | * | 2.25000 | * |
| 67 | 0 | 0.50372 | 1.08765 | 1.28408 | 1.60184 | 1.49948 | 1.49986 | 1.40041 |
| 71 | 0 | 0.50351 | 1.09729 | 1.39732 | 1.65384 | 1.62178 | 1.46927 | 1.65889 |
| 73 | 0 | 0.55852 | 1.25796 | 1.37843 | 1.58441 | 1.61481 | 1.43867 | 1.46433 |
| 79 | 0 | 0.50316 | 1.02226 | 1.40891 | 1.62967 | 1.56628 | 1.65248 | 1.64895 |
| 81 | 0 | 0.55556 | * | 1.41857 | 1.65924 | * | 1.56742 | 1.56742 |
| 83 | 0 | 0.50300 | 1.10991 | 1.37432 | 1.72373 | 1.44996 | 1.56494 | 1.61410 |
| 89 | 0 | 0.55300 | 1.19288 | 1.29922 | 1.51001 | 1.47747 | 1.53146 | 1.50130 |
| 97 | 0 | 0.55077 | 1.15420 | 1.40274 | 1.52626 | 1.46688 | 1.61483 | 1.65230 |