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Introduction

We start our presentation with a defense of abstract mathematics, highlighting the arguments provided in Journey Through Genius, and then broadening the inquiry by examining the contributions of G.H. Hardy's A Mathematician's Apology to this area.

We then provide some biographical information on Georg Cantor. He is credited with the creation of set theory, and the origination of the diagonalization proof of the non-denumerability of the real numbers, which is the Great Theorem of this section.

We describe the notions of infinity up to the 19th century and the central advancement that led to this finding: the use of bijections to establish cardinality of sets, both infinite and finite. We demonstrate this technique by evidencing bijection between the naturals and the integers and the naturals and the rationals, and a few additional miscellaneous examples.

We then present Cantor's diagonalization proof of the non-denumerability of the real numbers by showing that there is no bijection between the natural numbers and the real numbers in the open interval (0,1). We also evidence a bijection from the open interval (0,1) to the full real line, and thus demonstrate that the real numbers are not denumerable. We follow this with a discussion of the cardinality of the power set of the natural numbers. In particular, we lightly modify Cantor's proof to show that that the power set of the natural numbers is not denumerable.

We introduce the Cantor set, a delightful set that has uncountable members, but still manages to be of measure zero, and discuss a sly way of establishing its non-denumerability.

Finally, we make some parting comments pertaining to other legacies of Cantor, including the Axiom of Choice, and its equivalent statements, the Well Ordering Principal and Zorn's Lemma.

We close with a discussion on the beauty and durability of Cantor's proof of the non-denumerability of the real numbers.

On the Aesthetics of Mathematics

The 19th century was a time of increasing abstraction and generality in mathematics, which increased the breadth and depth of research in mathematics. This expansion was at least in part due to abandoning the previously held convention: that any mathematical problem needed to be (at least tangentially) tied to a physics or engineering problem. Mathematics had been a separate discipline, but it was one grounded in the observable.

Similarly, throughout this same time period, another set of human endeavors strayed toward the abstract: art. During the 19th century, there was a transition from the

Neoclassical's highly detailed renderings of realistic scenes to the Impressionistic emphasis on the movement of light in a scene that is abstractly (and quickly) rendered.

In both areas, detractors belittled the first halting steps toward independence. In art, the transition from art as strict depiction of a scene to a medium that conveyed the abstract was ultimately a supremely fruitful one. The subject fundamentally shifted from the particulars of the scene to the light, the color, or the feeling instilled by the scene.

Similarly, during the 19th century, there were entire branches of mathematics that were developed that didn't have any obvious tie to an existing problem. Alternative geometries were developed that included differing notions of the meaning of the term "parallel":

Euclid's 5th Postulate:

> *"That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."*

The parallel postulate of hyperbolic geometry:

> *"For any infinite straight line L and any point P not on it, there are many other infinitely extending straight lines that pass through P and which do not intersect L."*

And the parallel postulate of elliptic geometry:

> *"Through any point in the plane, there exist no lines parallel to a given line."*

These are mutually exclusive, and yet a large number of people devoted time to all of them. If one were to establish that the universe was actually one of these geometries, would the study of the alternate geometries be wasted?

The answer to that question is largely determined by what one views as the fundamental motivation for the study of mathematics. If one believed that the point was to make useful tools for other disciplines, then the answer would likely be "yes". If one was of the belief that the mathematics was an interesting and productive endeavor, irrespective of its utility in solving physics and engineering problems, then the answer would be "no".

Of course, there is an alternate possibility: perhaps mathematicians could dive into this sea of abstraction, under the assumption that eventually this abstraction would pay off by providing new methods to approach problems in physics and engineering. There are certainly a large number of instances where this occurred. In cosmology, one theory predicts that the universe globally complies with hyperbolic geometry; and that it is only the limitations of our perceptions that trick us into thinking the we live in a Euclidian world. Modern particle physics is accomplished using Lie Algebras. Perhaps much of today's "abstract mathematics" is tomorrow's "applied mathematics".

But is this relevant to our discussion? Do we demand that our Impressionistic paintings possess some utility outside of their aesthetic appeal? Are we plotting for future applications of Jackson Pollock's work? (Perhaps Pollock's techniques will find use in creating depictions of the results of an exploding paint truck, for instance!) This is absurd, of course. Art is treasured for its message, for its personal impact, for its aesthetic value. We don't expect to be able to club prey to death with the framed art, and similarly we should not base our assessment of the worth of a branch of mathematics by its projected possible uses!

The change in emphasis of mathematics resulted in abstraction, but not for its own sake. To quote G.H. Hardy (one of the most prominent mathematicians of the early 20th century), "It is not mere 'piling of subtlety of generalization upon subtlety of generalization' which is the outstanding achievement of modern mathematics." The ability to wander away from the well traveled paths led to the discovery of interconnections where none had been previously imagined.

In fact, Hardy had quite a lot to say on the aesthetic impact of mathematics; he wrote a thoroughly readable and poignant book on this particular topic. His argument is as follows:

Mathematics is "harmless".

Hardy argues that non-elementary mathematics, by its very abstract nature, is unlikely to be used for any direct mischief. This appealed to Hardy, who had at this point of his life had witnessed the destruction of World War I, and the genesis of World War II. He was happy to report that advanced mathematics did not find much utility in death dealing. (Along this line, it is also amusing to note that Paul R. Halmos's description of his dealings with the US army during this period seems to confirm that the day-to-day military machinations do not much cross paths with advanced mathematics; see chapter 7 of <u>I Want to be a Mathematician: An Automathography</u> for details.)

Mathematics is "eternal".

Portions of mathematics have lasted for thousands of years, and little else has. Further, the originators of much of this mathematics are still known to us by name. We still say "The Pythagorean Theorem" and "Euclid's Lemma", and still tell of Pythagoras drowning the mathematician (thought to be Hippasus) who proved that $\sqrt{2}$ is irrational. What's more, even in cases where the actual originator is unknown, we still have the mathematics created.

Mathematics helps to show the bounds of what humanity can accomplish.

Just as with mountain climbing, baseball, cricket, track and field, and blindfolded chess, doing mathematics helps to demonstrate the bounds of what can be accomplished by human-kind. In track and field, why is it relevant if an athlete can shave off a few milliseconds from a hundred yard dash? Nobody imagines that this accomplishment has direct utility in our day to day lives, but it does give us some sense for how fast humans can run, and that's something that all humans can be proud of. I couldn't run that quickly, but someone who isn't that distantly related to me can, and so I have some connection to that accomplishment. Similarly, I may not be able to think as deeply as Grigori

Perelman, but I can appreciate (at least some of) the complexities of the Poincaré Conjecture.

And if mathematics is without true merit...

Even if the study of abstract mathematics is ultimately seen to be without merit, its study isn't that wasteful. It may be a different matter entirely if the study of mathematics was a significant drain on the world economy, but it occurs at such a small scale that there simply isn't much to be lost. Though it is true that there are more mathematicians active today than have ever lived, the percentage of the world's population dedicated to research into mathematics is simply too small to be significant.

Mathematics is beautiful.

This final point is of vital importance. It is this point which establishes mathematics as an aesthetic discipline. Just as with art that is hung in a museum, not everyone will see its beauty, but that doesn't imply that it isn't beautiful, simply not widely appreciated. Paul Erdös said it well: "Why are numbers beautiful? It's like asking why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is."

Hardy phrased it as such: "A mathematician, like a painter or a poet, is a maker of patterns… The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics."

Notions of Infinity to the 19th Century

The results that we deal with principally deal with the nature of infinity. Up to this time, the notion of infinity had undergone some development, but was still poorly understood. From Ancient Greek times, it was clear that the notion of infinity was viewed with some skepticism. Zeno's paradoxes (ca. 450 BC) can be viewed as a protracted argument against the logical consistency of infinity.

This notion had advanced somewhat by the time of Antiphon the Sophist (ca. 410 BC), who developed a notion of finding the areas of shapes through the method of exhaustion, a style of limiting argument where the area of an object was obtained by summing inscribed regular polygons, and finally deriving the area through a proof by contradiction. Archimedes (ca. 250) again used a related method to estimate the area and circumference of a circle.

In their development of calculus, both Newton and Leibniz brushed up against infinity, but neither really attempted to formalize it. Newton's notion of ratios whose numerator and denominator both approach 0 but didn't quite every reach 0 shouts limits to us today, but was viewed as strongly connected to the infinite at the time. Leibniz's notion of undividable infinitesimals is still taught today when attempting to convey the notion of integration, and was even formalized by Robinson's Nonstandard Analysis in the 1960s.

Finally, both Gauss and Cauchy ultimately viewed infinity as a symbol rather than a distinct number; an indication that a function of a sequence grows to be unbounded.

Cantor solidified aspects of infinity, and ultimately demonstrated that there are many styles of infinity.

Biography

Georg Ferdinand Ludwig Philipp Cantor was born in 1845 in Copenhagen, Denmark. Eldest of six children, Georg was raised to be a devout Lutheran.

In 1856, when his father became ill, the family moved to Wiesbaden and later Frankfurt, seeking milder winters. In 1860, Cantor graduated with distinction from the Realschule in Darmstadt, where his exceptional skills at mathematics were noted. His father wanted Georg to study engineering, and initially he did, but eventually he convinced his father that he should be allowed to study mathematics. So in 1862, following his father's permission, Cantor entered the Federal Polytechnic Institute in Zurich and began studying mathematics.

After his father's death in 1863, Cantor shifted his studies to the University of Berlin, studying under Weierstrass, Kummer, and Kronecker, and befriended fellow student Hermann Schwarz. In 1867, Berlin granted him the Ph.D. for a thesis on number theory.

After teaching for a year in a Berlin girl's school, Cantor took up position at the University of Halle, where he spent his entire career.

Cantor married in 1874, having six children by 1886. Thanks to an inheritance from his father, Cantor was able to support the family despite his modest academic pay.

Cantor was promoted to Extraordinary Professor in 1872 (and thus started drawing a salary), and made full professor by the age of 34, a notable achievement. But what Cantor wanted was a chair at a more prestigious university, in particular at Berlin, then the leading German university. However, Kronecker, who headed mathematics at Berlin, and Cantor's old friend Hermann Schwarz, who now worked at Berlin, were not interested in having him as a colleague.

Worse still, Kronecker fundamentally disagreed with the thrust of Cantor's work. Kronecker is now seen as one of the founders of the Constructivist movement in mathematics, a view that it is necessary to construct a mathematical object in order to prove that it exists. Cantor came to believe that, due to Kronecker's stance and influence, that he would never leave Halle.

In 1881, a chair at Halle opened, and they accepted Cantor's suggestion that it be offered to Dedekind, Heinrich Weber, and Franz Mertens, in that order, but each declined the chair after being offered it, highlighting Halle's lack of standing among German mathematics departments.

In 1884, Cantor suffered his first bout of depression. The emotional crisis this created lead Cantor to explore other areas of study, most notably philosophy and literature. It was during this period that he first started to explore his theory that the works attributed to William Shakespeare were actually written by Francis Bacon. During this time he wrote fifty-two letters to Mittang-Leffler, all of which attacked Kronecker.

He eventually recovered, but never regained the high level of his earlier work. He eventually sought reconciliation with Kronecker, who graciously accepted, and was surprised that there was any rancor between them. Nevertheless, Kronecker's objections to Cantor's work persisted.

It was once thought that Cantor's recurring bouts of depression were caused by this opposition to his work. While it's likely that Cantor's mathematical worries and his difficulties dealing with certain people were magnified by his depression, it is doubtful whether they were its cause; Cantor is currently thought to have be afflicted with bipolar disorder.

After his youngest son's death in 1899, Cantor suffered from chronic depression for the rest of his life, for which he was excused from teaching and repeatedly confined in various sanatoria. He did not give up on mathematics completely, lecturing on the paradoxes of set theory at a meeting of the German mathematical society.

Cantor retired in 1913, and died in 1918, having spent the last year of his life in a sanatorium begging his wife to be released.

Cantor's Work

Cantor's first ten papers were on number theory; after that his work turned to analysis at the suggestion of Hiene, one of his colleagues at Halle. His first great work was solving the open problem that had eluded Hiene, the existence and uniqueness of a representation of a function by trigonometric series. Between 1870 and 1872, Cantor did more work with trigonometric series, including defining irrational numbers as a convergent sequence of rational numbers. Dedekind would later cite Cantor in his work defining the real numbers with his celebrated "Dedekind cuts".

Cantor is credited with the creation of set theory during this time. His creation of sets, and examining the number of elements in these sets arose naturally enough from his research into trigonometric series. Cantor had discovered that the number of discontinuities of the function being represented as a trigonometric series determined whether this series converges, and was unique. Cantor formed sets of these points of discontinuity, and examined the set's size.

Cantor envisioned sets as arbitrary collections of items, and his framework allowed any arbitrary group of items to be included, even if the description yielded inconsistencies. In just a few years, Bertrand Russell's paradox would suggest that this allowance was overly broad: that it would allow for the creation of logically inconsistent sets. In response, this notion of set theory would go on to be refined: Zermelo-Frankel set theory would disallow making sets purely through arbitrary description, explicitly disallowing sets like "the set of all sets". von Neumann-Bernays-Gödel set theory refers to such collections as "classes" and more manageable sub-portions "sets".

The fundamental problem is that frameworks that are too broad allow for inconsistent statements to be made, a phenomenon that should be familiar to English speakers everywhere. This same property of English allows for the formation of meaningless, but syntactically valid sentences like "Is the designated hitter rule more or less green than the fatness of a pig?" (A suggested answer to this question is "The Treaty of Paris").

Cantor's contribution wasn't just "throwing stuff into sets". More significantly, he developed a method for finding (cardinality) equivalence between two sets. He proposed the use of the bijection, that is, a function which is both onto (surjective) and one-to-one (injective).

This is clearly reasonable in the setting of finite sets, but it is also vacuous in this setting (where counting works just as well). The use of this characterization, however, becomes extraordinarily important in dealing with sets that are not finite.

On the Cardinality of Infinite Sets

Some easy, but conceptually odd, examples first. A bijection between the odd integers and the even integers isn't surprising, as it's clear that the two sets ought to be comparable. For the purpose of example, one such bijection is this:

$\phi:\{2n:n\in\mathbb{Z}\}\xrightarrow[onto]{1-1}\{2n+1:n\in\mathbb{Z}\}$ given by $\phi(k)\to k+1$. We just associate each even number with the odd number to its right.

We shall initially be interested in sets that can be put into a bijective correspondence with the set of natural numbers (that is, the positive integers). Recalling our proof of the infinitude of primes, we can evidence a fairly counter-intuitive bijection:

$\phi:\mathbb{N}\xrightarrow[onto]{1-1}\{p:p\in\mathbb{N},\ \text{p is prime}\}$. This seems quite odd, as the "number" of primes appears to be much less than the number of other integers. Other measures of the density of primes suggest that our intuition is incorrect. Note, for example, that the series $\sum_{p\ \text{prime}}\frac{1}{\text{p}}$ diverges, which suggests that the primes are fairly dense! An uninteresting bijection from the natural numbers to the primes is as follows: $\phi(n)=p_n$, where p_n is simply the nth prime.

Any set that can be put into a bijective relationship with the natural numbers is called "countably infinite" or "denumerable". The cardinality of the natural numbers is written $\aleph_0$. Further investigation of these sets is aided by two simple lemmas:

Lemma 1: The cardinality of the (disjoint) union of a countably infinite set and a finite set is countably infinite.
Proof: Let us call our countably infinite set A and our finite set B. By hypothesis, B has a finite number of members, which we'll call b_1, b_2, ..., b_m. We'll define the bijection

$\phi : \mathbb{N} \xrightarrow[onto]{1-1} A \cup B$ as $\phi(n) = \begin{cases} b_n & n \le m \\ a_{n-m} & n > m \end{cases}$

(that is, we'll enumerate the finite set with the first m terms, and then enumerate the countably infinite set.).

Lemma 2: The cardinality of the (disjoint) union of two countably infinite sets is countably infinite.

Proof: Let us call our sets A and B. We'll define the map $\phi : \mathbb{N} \xrightarrow[onto]{1-1} A \cup B$

$\phi(n) = \begin{cases} a_{\frac{n}{2}} & \text{n is even} \\ b_{\frac{n+1}{2}} & \text{n is odd} \end{cases}$, which is also a bijection.

Corollary 1: The integers are countable.
Proof: The set of positive integers is countable (the bijection is the identity), as is the set of negative integers (the bijection is negative one multiplied by the identity). We are left with the set containing only 0, a finite set. So, we have the disjoint union of two countably infinite sets (which is countably infinite by lemma 2), and a finite set, yielding a countably infinite set.

Theorem: The rational numbers are countably infinite. We'll define a way of enumerating the positive rational numbers. We'll setup a table where the numerator increases as we move right in the table, and the denominator increases as we move down.

$$\begin{matrix} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \cdots \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \cdots \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{matrix}$$

We need to exclude the numbers that we have previously included in order to obtain the injective property. We clearly will have enumerated all of $(a,b) \in \mathbb{N} \times \mathbb{N}$ possibilities, and we have excluded equivalent rational numbers by formation of the map.

Thus, we have a bijection $\phi : \mathbb{N} \to \mathbb{Q}^+$. We can similarly make a bijection from $\phi^- : \mathbb{N} \to \mathbb{Q}^-$ by just multiplying each entry in the table by negative 1. Thus, both $\mathbb{Q}^+$

and $\mathbb{Q}^-$ are countable, and we have only excluded 0. As with the integers, we can take the union of all three of these sets, and we are left with a countable set.

A similar argument can be made regarding the algebraic numbers (that is, the numbers which are roots of polynomials with integer coefficients) by noting that each polynomial has only finite roots, and then by arranging the polynomial into towers in a similar way. Thus the algebraic numbers are also countable.

The Great Theorem

Theorem: The open interval $(0,1)$ is not denumerable.

Proof:

We'll assume the contrary, that is, we assume that there is a bijection between the open interval $(0,1)$ and the natural numbers.

As this set is assumed to be countable, it can be enumerated. We'll enumerate this set as a sequence of infinite decimals (we'll demand that all decimals that could terminate, do terminate, with a tail of of 0s), and we'll arrange these in a tabular form.

We'll produce a number, b, that could not possibly be in this list by moving down the principal diagonal of this list, each time choosing a number different than the current place that is different than 0 or 9. (we exclude 0 and 9 to prevent any ambiguity that results from repeating decimal values).

$$
\begin{array}{cccccccc}
0 & . & 1 & 0 & 2 & 1 & 1 & \ldots \\
0 & . & 0 & 0 & 2 & 0 & 1 & \ldots \\
0 & . & 1 & 2 & 3 & 4 & 5 & \ldots \\
0 & . & 3 & 2 & 8 & 1 & 1 & \ldots \\
0 & . & 5 & 2 & 8 & 9 & 1 & \ldots \\
0 & . & 3 & 6 & 5 & 9 & 3 & \ldots \\
0 & . & 9 & 7 & 0 & 4 & 5 & \ldots \\
 & & & & \vdots & & &
\end{array}
$$

The resulting number b is a real number, because it is an infinite decimal. Because b cannot be 0.999999… or 0.000000… it cannot be 0 or 1. Therefore b is strictly between 0 and 1. Because b is strictly between 0 and 1, it must fall somewhere on our list of real numbers, but we have specifically constructed b such that b will not appear on our list. If it were the nth element of the list, then the nth decimal entry will have a different value (and thus it could not be equal to any of them)..

Thus b cannot be an element of our list, and we have reached a contradiction, so our assumption is false. Therefore the real numbers between 0 and 1 are not denumerable.

Theorem: The real numbers are of the same cardinality as the numbers in the set $(0,1)$. We can use any rational expression of the correct form to show this. We'll examine $f(x)=\dfrac{2x-1}{x(1-x)}$. This clearly has a root at $x=\dfrac{1}{2}$, and vertical asymptotes at 0 and 1. Examining the derivative of our function, we find that it is strictly positive in the interval $(0,1)$, so we have a one-to-one function whose range is $(-\infty,\infty)$. Thus, on the domain $(0,1)$ we have a bijection: $f:(0,1)\xrightarrow[onto]{1-1}\mathbb{R}$, so the cardinalities of these two sets are equivalent.

Corollary: The real numbers are not countable.

Diagonalization Miscellanea

We can use this same argument with binary strings, viewing these binary strings as infinite binary expansions of numbers in this same interval, and arrive at the same contradiction (but with a few minor issues having to do again with ambiguity imposed by a tail of repeating 1s). We'll view this in different way that removes this issue. Instead, think of this sequence of 1s and 0s as a sequence of Boolean values, each of which establishes if a particular element of the natural numbers is present or not present (where each place corresponds to exactly one natural number).

Each of these strings can be viewed as a description of one possible subset of the natural numbers, and every possible subset is ultimately enumerated. Through the same diagonalization argument, we can establish that the power set of the natural numbers is not the same cardinality as the natural numbers. This holds in general, but is the topic of the next section.

In fact, the Diagonalization problem has significant applications beyond set theory. The Halting Problem from computer science (which asks if an arbitrary Turing machine with finite state information will halt, given a finite length program), can be shown reduce to a diagonalization argument! In this setting, this shows that the Halting problem is undecidable, and thus there is no algorithm for making this determination.

More Set Theoretic Fun

We can use the tools provided to examine some rather unusual sets. We can define the Cantor set by starting with the closed interval $[0,1]$, and removing a sequence of "open thirds". More formally:

Let $c_1=[0,1]\setminus\left(\dfrac{1}{3},\dfrac{2}{3}\right)$, $c_2=c_1\setminus\left(\left(\dfrac{1}{9},\dfrac{2}{9}\right)\cup\left(\dfrac{7}{9},\dfrac{8}{9}\right)\right)$, etc. The Cantor set is $C=\lim\limits_{n\to\infty}c_n$.

One can establish that this set is non-empty by following the endpoints of the closed intervals. Further, one can sum the lengths of the removed intervals, resulting in a geometric sum that converges to 1, implying that we have, in some sense, removed all the content. This argument can be formalized by appealing to some measure theory, upon which we find that the Cantor set is of measure zero.

Finally, one can make a map from $\psi : C \rightarrow \{0,1\}^{\infty}$, where we take any point from the Cantor set and categorize it based on whether it lies to the left or the right of that round's removed interval. Thus, each point can be uniquely described by a sequence of "left/right" decisions, which correspond to "0/1" digits, respectively. Thus, we know that the Cantor set's cardinality is at least as large as the real numbers, by the subset argument previously. Further, the Cantor set is a proper subset of the real numbers, so we know that the cardinality of the real numbers is at least as large as the cardinality of the Cantor set. Thus, we know that the cardinality of the Cantor set is the same as the cardinality of the real numbers! Thus have a measure-zero subset of the real numbers which is uncountably large!

The Axiom of Choice

The axiom gets its name not because mathematicians prefer it to other axioms.
— A. K. Dewdney

The Axiom of Choice is an interesting aside regarding set theory.

Definition: Let *C* be a collection of nonempty sets ($\varnothing \notin$ C). A choice function, f, is a function such that for all X $\in$ S, f(X) $\in$ X. (Intuitively, we can *choose* a member from each set in that collection.)

Axiom of Choice (Axiom of Choice): Every family of nonempty sets has a choice function.

The Axiom of Choice was formulated by Zermelo in 1904. This axiom is non-constructive, and it guarantees the existence for a choice function, but gives no indication how to make such a function. Because of the non-constructive nature of the axiom and some of the fairly non-intuitive results of the axiom, it didn't gain broad acceptance until quite recently. In 1940, Kurt Gödel proved that (as long as the pre-existing axioms were without contradiction) adding the axiom of choice did not lead to a contradiction with the axioms set theory. In 1963, Paul Cohen demonstrated that adding the negation of the axiom of choice to the axioms of set theory also leads to no contradiction (assuming, once again that the axioms were consistent to begin with), thus the Axiom of Choice is independent of Zermelo-Fraenkel (ZF) set theory.

In general, the Axiom of Choice is required in order to make arbitrary choices from a family of sets. There are a few specific instances where the Axiom of Choice is not necessary to accomplish this task:

- If each set in the family is a singleton.
- If there are only finite sets in the family (induction on the number of sets in the family suffices to show that selection can occur for any finite number of sets)

- If each X $\in$ S contains only a finite number of ordered (distinguishable) items, (e.g., f(X) = the least element of a finite set)

Russell phrased it as this: If we have $\aleph_0$ null pairs of shoes, then we can select one shoe from each pair without the axiom of choice (just choose the left shoe for each pair). But, if we had $\aleph_0$ pairs of socks, then we need the Axiom of Choice to pick one from each set (because socks are not distinguishable from each other).

The Axiom of Choice has many equivalent statements. A few of the most compelling are:

- For any relation R there is a function F $\subseteq$ R, with domain(F) = domain(R)
- The Cartesian product of a non-empty family of non-empty sets is non-empty.
- For any two sets C and D, C $\preceq$ D or D $\preceq$ C (i.e., the cardinality of any two sets is comparable)

Banach-Tarski Paradox: Using the Axiom of Choice, it is possible to take the 3-dimensional closed unit ball, and partition it into finitely many pieces, and move those pieces in rigid motions (i.e., rotations and translations, with pieces permitted to move through one another) and reassemble them to form two copies of B. (Note, the pieces of the ball are not Lebesgue measurable.)

The Axiom of Choice is also equivalent to a very non-intuitive theorem, known as the Well-ordering theorem. For this theorem, we'll need some additional definitions.

Definition A binary relation, < , is a *total-ordering* of a set S if for all p, q, r in S:

$p \nless p$ (< is irreflexive)

If p < q and q < r then p < r (< is transitive)

For all p, q in S, p < q or p = q or q < p

A binary relation < is a *well-ordering* of a set S if

(S, <) is a total-ordering.

Every subset of S has a least element

Definition An element $a \in X$ is the *least element* of X if $(\forall x \in X)\, a \le x$

Any subset of the positive integers is a well-ordered under the standard "<" operator (by the Well Ordering Principle of the integers).

Zermelo's Well Ordering Theorem: *Every set can be well ordered.*

This appears to contradict a fundamental characteristic of open intervals on the real line (that there is no minimal element by our standard less than ordering), but it does not. Sadly, the reason that it does not is that we are told that an ordering exists, but we are not told that this ordering will actually, in any way, reflect our notions on the structure of the real line. That is to say, the ordering could well be utterly without use other than to win arguments on pedantic grounds.

Another statement that is equivalent to the Axiom of Choice is called Zorn's Lemma:

Zorn's Lemma: If $(X, <)$ is a nonempty partially ordered set such that every chain (totally ordered subset of X) in X has an upper bound, then X has a maximal element.

Zorn's lemma (and thus the Axiom of Choice) is used to prove several well known results:

- Every vector space has a basis
- Every field has a unique algebraic closure
- *Tychonoff's Theorem*: Any product of compact topological spaces is compact (Note: for compact Hausdorff spaces Tychonoff's Theorem is equivalent to the Boolean Prime Ideal Theorem (Rubin and Scott 1954) and hence weaker than the Axiom of Choice.)

More generally, the Axiom of Choice is used to prove many other familiar results:

- The countable union of countable sets is countable.
- The Baire Category Theorem (a weakened version of Axiom of Choice is required here: The Axiom of Dependent Choice)
- Every infinite set has a denumerable subset

The generalized continuum hypothesis (GCH) is not only independent of ZF, but also independent of ZF plus the axiom of choice (ZFC). However, if one assumes the the ZF axioms and GCH, then one can derive the Axiom of Choice, making GCH a strictly stronger claim than Axiom of Choice.

Q: What's sour, yellow, and equivalent to the axiom of choice?
A: Zorn's lemon.

Conclusion
Why is this proof beautiful? There are many reasons; it fulfills all of Hardy's criteria for beauty:
The diagonalization proof is an easy, elegant proof. This makes it an easy proof to teach, and an easy proof to understand.
It is non-obvious. Why should there be different types of infinity, after all?

Why has this proof endured? Well, the principal reason that must be noted is that it hasn't endured for very long yet. It's only a bit over 100 years old at this point. But we do suspect that it will endure as it is easy and surprising, and it has a significant impact on many diverse areas of mathematics. Almost any area of mathematics that gets coupled with set theory, and which uses infinite sets, runs into these notions.

"The fear of infinity is a form of myopia that destroys the possibility of seeing the actual infinite, even though it in its highest form has created and sustains us, and in its secondary transfinite forms occurs all around us and even inhabits our minds. "
Georg Cantor

References

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Homework for Chapter 11

1) What is Hardy's basic argument that mathematicians should be allowed to do mathematics.

2) Produce a bijection between the even and odd numbers.

3) Produce a bijection between $(0,1)$ and $\mathbb{R}$ that is different than the one discussed in class.

4) What is the danger of allowing the choice of 0 or 9 in Cantor's digitalization proof?

5) The Cantor Set can be defined by starting with the closed unit interval $[0.1]$, and continuing to remove middle third of the remaining closed intervals. More formally: Let $c_1 = [0,1] \setminus \left(\frac{1}{3},\frac{2}{3}\right)$, $c_2 = \left(c_1 \setminus \left(\frac{1}{9},\frac{2}{9}\right)\right) \setminus \left(\frac{7}{9},\frac{8}{9}\right)$, etc. The Cantor set is $c_\infty = \lim_{n\to\infty} c_n$.

a) Argue that the Cantor set is not empty.

b) What is the size of the removed portion of the Cantor set? (Hint, consider summing the lengths of the removed portions.)

Extra Credit:
What is the cardinality of the Cantor set, and why?