Math 2J Key

Quiz 4

1) $\sum_{k=2}^{\infty} \frac{4}{k^2-1}$

a. What is an equation for the nth partial sum for this series? (Hint: use partial fraction decomposition then write out several terms and see how they cancel)

$$\frac{4}{k^2-1} = \frac{4}{(k-1)(k+1)} = \frac{A}{k-1} + \frac{B}{k+1} = \frac{A(k+1)+B(k-1)}{(k-1)(k+1)} = \frac{k(A+B)+(A-B)}{(k-1)(k+1)}$$

$A-B=4$ and $B=-A$ so $A=2$ and $B=-2$ so

$$\sum_{k=2}^{\infty} \frac{4}{k^2-1} = \sum_{k=2}^{\infty}\left(\frac{2}{k-1} - \frac{2}{k+1}\right) = 2\sum_{k=2}^{\infty}\left(\frac{1}{k-1} - \frac{1}{k+1}\right)$$

We are looking for $s_n = \sum_{k=2}^{n} \frac{4}{k^2-1} = 2\sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{1}{k+1}\right)$. Expanding for $n=6$,

$s_6 = 2\left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{6} - \frac{1}{8}\right)\right]$ which suggests that the general form is

$$s_n = 2\left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \dots + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right)\right]$$

$$s_n = 2\left[\frac{1}{1} + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right] = 3 - \frac{2}{n} - \frac{2}{n+1}$$

b. Does this infinite series converge? If so, what does it converge to?

$s_n = 3 - \frac{2}{n} - \frac{2}{n+1}$ so $\sum_{k=2}^{\infty} \frac{4}{k^2-1} = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(3 - \frac{2}{n} - \frac{2}{n+1}\right) = 3 - \lim_{n\to\infty}\left(\frac{2}{n} + \frac{2}{n+1}\right) = 3 - 0 = 3$

so $\sum_{k=2}^{\infty} \frac{4}{k^2-1} = 3$.

2) $\sum_{k=0}^{\infty} 3^k$

a. What is an equation for the nth partial sum for this series?

$s_n = \sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$ so $s_n = \sum_{k=0}^{\infty} 3^k = \frac{1-3^{n+1}}{1-3} = \frac{3^{n+1}-1}{2}$

b. Does this infinite series converge? If so, what does it converge to?

$\sum_{k=0}^{\infty} 3^k = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{3^{n+1}-1}{2}\right) = \infty$ so $\sum_{k=0}^{\infty} 3^k$ diverges to ∞.

3) Prove either 1b or 2b using the relevant definition of sequential convergence or divergence.

For 1b Our assertion was that $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(3-\frac{2}{n}-\frac{2}{n+1}\right)=3$. To prove this we need to show that for any $\varepsilon > 0$ there is an integer N such that if $n > N$ then $|s_n - 3| < \varepsilon$. To find such an N, we solve:

$|s_n - 3| = \left|\left(3-\frac{2}{n}-\frac{2}{n+1}\right)-3\right| = \left(\frac{2}{n}+\frac{2}{n+1}\right) < \varepsilon$. We'll avoid some unfortunate algebra by noting that $\left(\frac{2}{n}+\frac{2}{n+1}\right) < \left(\frac{2}{n}+\frac{2}{n}\right)$, so if we can show that $\frac{4}{n} < \varepsilon$ then $\left(\frac{2}{n}+\frac{2}{n+1}\right) < \frac{4}{n} < \varepsilon$. If $n > \frac{4}{\varepsilon}$ we get this last inequality.

So, if we choose $n > N \ge \frac{4}{\varepsilon}$ then $|s_n - 3| = \frac{2}{n}+\frac{2}{n+1} < \frac{4}{n} < \frac{4}{\left(\frac{4}{\varepsilon}\right)} = \varepsilon$ so $|s_n - 3| < \varepsilon$ as desired. This proves that $\lim_{n\to\infty} s_n = 3$.

For 2b Our assertion is that $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{3^{n+1}-1}{2}\right) = \infty$. To prove this we need to show that for any bound $M > 0$ that there is an integer N such that if $n > N$ then $s_n > M$. To find such an N, we solve:

$\frac{3^{n+1}-1}{2} > M$ so $3^{n+1}-1 > 2M$ and $3^{n+1} > 2M+1$ and $n+1 > \log_3(2M+1)$ so $n > \log_3(2M+1)-1$. So, if we choose $n > N \ge \log_3(2M+1)-1$ then

$s_n = \frac{3^{n+1}-1}{2} > \frac{3^{[\log_3(2M+1)-1]+1}-1}{2} = \frac{3^{\log_3(2M+1)}-1}{2} = \frac{(2M+1)-1}{2} = \frac{2M}{2} = M$ so $s_n > M$ as desired. This proves that $\lim_{n\to\infty} s_n = \infty$.