If you have a second order Cauchy-Euler differential equation
\[ a_2 x^2 y'' + a_1 x y' + a_0 y = g(x) \]
you can convert this equation into a linear ODE with constant coefficients by making a substitution: \( x(t) = e^t \). We can undo this substitution using \( t(x) = \ln x \). It's clear how to use this substitution to get rid of the \( x \) terms, but what of the derivatives?

We start by noting that this differential equation is all in terms of \( x \):
\[ a_2 x^2 \frac{d^2 y}{dx^2}(x) + a_1 x \frac{dy}{dx}(x) + a_0 y(x) = g(x) \]

We want to convert our differential equation so that it is in terms of \( t \) rather than \( x \). To do this, let's define a new version of the solution:
\[ \tilde{y}(t) = y(x(t)) = y(e^t) \]

Our new differential equation will be in terms of \( \tilde{y}(t) \) and its derivatives (which will be with respect to \( t \)). With this new version of the function, we can represent our original function:
\[ y(x) = \tilde{y}(t(x)) = \tilde{y}(\ln x) \]

We can use these new functions to find out what \( \frac{dy}{dx} \) and \( \frac{d^2 y}{dx^2} \) in terms of \( \tilde{y}(t) \) and its derivatives:
\[ \frac{dy}{dx} = \frac{d}{dx} \left[ y(x) \right] = \frac{d}{dx} \left[ \tilde{y}(\ln x) \right] = \frac{d\tilde{y}}{dt} \frac{1}{x} = e^{-t} \frac{d\tilde{y}}{dt} \]
(this is an application of the chain rule)

We use this first derivative to calculate the second derivative:
\[ \frac{d^2 y}{dx^2} = \frac{d}{dx} \left[ \frac{1}{x} \tilde{y}'(\ln x) \right] = \frac{d}{dx} \left[ \frac{1}{x} \right] \tilde{y}'(\ln x) + \frac{1}{x} \frac{d}{dx} \left[ \tilde{y}'(\ln x) \right] \]
\[ = -\frac{1}{x^2} \frac{d\tilde{y}}{dt} + \frac{1}{x^2} \frac{d^2 \tilde{y}}{dt^2} \]
\[ = e^{-2t} \left( \frac{d^2 \tilde{y}}{dt^2} - \frac{d\tilde{y}}{dt} \right) \]
For higher order equations, you can continue to apply this same process to find the higher derivatives.

Summarizing:

\[
\frac{dy}{dx} = e^{-t} \frac{d\tilde{y}}{dt}
\]

and

\[
\frac{d^2 y}{dx^2} = e^{-2t} \left( \frac{d^2 \tilde{y}}{dt^2} - \frac{d\tilde{y}}{dt} \right)
\]

Notice that these didn't depend on the particular differential equation; these work any time that you use this substitution.

In the abstract, we can apply this directly to the general form:

\[
a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = g(x) \xrightarrow{\text{substitution}} a_2 e^{2t} \left[ e^{-2t} \left( \frac{d^2 \tilde{y}}{dt^2} - \frac{d\tilde{y}}{dt} \right) \right] + a_1 e^t \left[ \frac{d\tilde{y}}{dt} e^{-t} \right] + a_0 y(e') = g(e')
\]

\[
\Rightarrow a_2 \frac{d^2 \tilde{y}}{dt^2} + (a_1 - a_2) \frac{d\tilde{y}}{dt} + a_0 \tilde{y} = g(e') \rightarrow a_2 \tilde{y}'' + (a_1 - a_2) \tilde{y}' + a_0 \tilde{y} = g(e')
\]

So, one way to approach this class of problem is just to remember that the substitution yields:

\[
a_2 x^2 y'' + a_1 xy' + a_0 y = g(x) \xrightarrow{\text{substitution}} a_2 \tilde{y}'' + (a_1 - a_2) \tilde{y}' + a_0 \tilde{y} = g(e')
\]

This ODE now has constant coefficients, and can thus be approached by our standard methods. Once we have a solution, \(\tilde{y}(e')\), we can apply the reverse substitution to get a solution for our original ODE:

\[
y(x) = \tilde{y}(\ln x)
\]
As an example, look at the differential equation:

\[ x^2 y'' + 10xy' + 8y = x^2 \]

Now, substitute into the differential equation (using what we found above!):

\[
e^{2t} \left[ e^{-2t} \left( \frac{d^2 \tilde{y}}{dt^2} - \frac{d\tilde{y}}{dt} \right) \right] + 10e^t \left( \frac{d\tilde{y}}{dt} e^{-t} \right) + 8\tilde{y}(t) = e^{2t}
\]

Simplifying:

\[
\tilde{y}'' + 9\tilde{y}' + 8\tilde{y} = e^{2t}
\]

Alternately, note that

\[
a_2 x^2 y'' + a_1 xy' + a_0 y = g(x) \rightarrow a_2 \tilde{y}'' + (a_1 - a_2) \tilde{y}' + a_0 \tilde{y} = g(e^t),
\]

so

\[
x^2 y'' + 10xy' + 8y = x^2 \rightarrow \tilde{y}'' + 9\tilde{y}' + 8\tilde{y} = e^{2t}
\]

At this point, we can solve using the standard methods. The differential operator here is

\[ L = D^2 + 9D + 8 \]

The auxiliary equation for the homogeneous case is

\[ m^2 + 9m + 8 = (m + 1)(m + 8) = 0 \]

so

\[ \tilde{y}_c(t) = c_1 e^{-t} + c_2 e^{-8t} \]

We guess that a particular solution could have the form \( \tilde{y}_p(t) = Ae^{2t} \). Applying the differential operator:

\[ L\tilde{y}_p = 4Ae^{2t} + 18Ae^{2t} + 8Ae^{2t} = 30Ae^{2t} \]

This is supposed to equal to \( e^{2t} \), so \( A = \frac{1}{30} \), resulting in our particular solution

\[ \tilde{y}_p(t) = \frac{1}{30} e^{2t} \]

\[ \tilde{y}(x) = \tilde{y}_c(x) + \tilde{y}_p(x) \] so our solution is

\[ \tilde{y}(t) = c_1 e^{-t} + c_2 e^{-8t} + \frac{1}{30} e^{2t}. \]

Now, reverse the substitution:

\[ y(x) = \tilde{y}(\ln x) = c_1 e^{-\ln x} + c_2 e^{-3\ln x} + \frac{1}{30} e^{2\ln x} \]

or:

\[ y(x) = c_1 \frac{1}{x} + c_2 \frac{1}{x^8} + \frac{1}{30} x^2 \]