Divergence of the Harmonic Series

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A wonderful proof for the divergence of the Harmonic series proceeds by manufacturing a related positive series that is strictly smaller than the harmonic series, but which diverges. The comparison test then tells us that the harmonic series must also diverge.

To build this related series, we'll take the harmonic series and group terms into expressions that we can bound.

The harmonic series is:

$$\sum_{j=1}^{\infty}\frac{1}{j}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\cdots$$

We group together terms by powers of 2, that is, we'll group the first term by itself ($2^0=1$), then we'll have a group of 2 terms ($2^1=2$), then a group of 4 terms, then a group of 8 terms, etc.

$$\sum_{j=1}^{\infty}\frac{1}{j}=\left(\frac{1}{1}\right)+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\left(\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}\right)+\cdots$$

Now note that each term in the series strictly decreases, so we can say that every term in a grouping is larger than the smallest term in the next grouping. This is the key insight in making our new series, which will be after the inequality:

$$\begin{aligned}\sum_{j=1}^{\infty}\frac{1}{j}&=\left(\frac{1}{1}\right)+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\left(\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}\right)+\cdots\\&\geq\left(\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}\right)+\cdots\end{aligned}$$

Each grouping has a power of two terms, and each denominator in our new series is also a power of two, so now we have

$$\sum_{j=1}^{\infty}\frac{1}{j}\geq\sum_{j=1}^{\infty}\left(2^{i-1}\left(\frac{1}{2^i}\right)\right)=\sum_{j=1}^{\infty}\frac{1}{2}$$

Our new series clearly diverges, so the harmonic series diverges by the comparison test.