An Elementary Recurrence Relation for π

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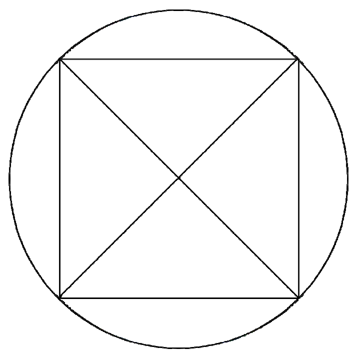
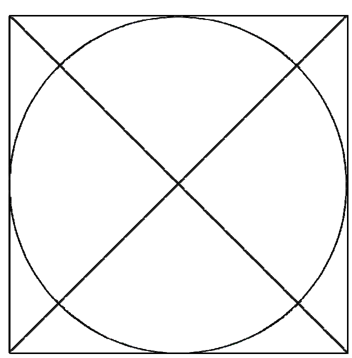
π is defined to be the ratio of the circumference of a circle and its diameter. It's difficult to actually estimate this value using honest-to-goodness circles, given their... uh... bendiness.

Archimedes decided that it would be reasonable to take a circle, and inscribe a regular n-gon (that is, draw the n-gon so that its vertices are on the circle; the term "regular" means that each of its sides is the same length.), and then take the circumference from that n-gon (which is relatively easy to figure out) and then divide by the diameter of the circle. For example, a regular 4-gon is a square, depicted below inscribed in a circle.



Now, because we've inscribed the shape, the circumference of our n-gon is smaller than that of the circle (this seems intuitively obvious, but can be proven rigorously by applying the triangle inequality) so when we divide by the diameter of the circle, we get a *lower bound* for π. As we increase the number of sides of our n-gon, we get less and less "slop", and our polygon's circumference grows closer and closer to the circumference of an actual circle, and our estimated value for π increases.

Now, look at it from the opposite end: Let's circumscribe a regular n-gon, where each face of the n-gon only touches the circle once. For example, below we have a square circumscribing the same circle.

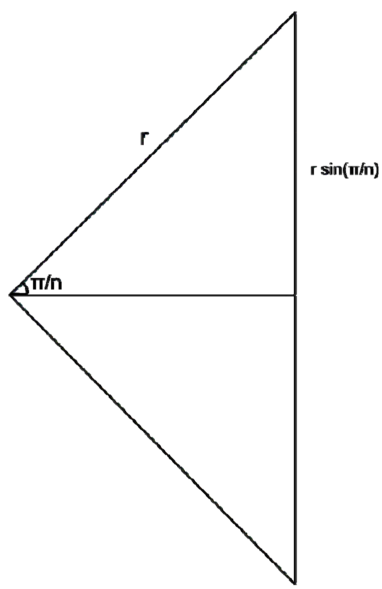


It ends up that this new circumscribed n-gon will provide an *over estimate* for the circumference, so if we divide by the diameter of the inscribed circle, we'll get an upper bound for π. We'll show that this is an upper bound shortly.

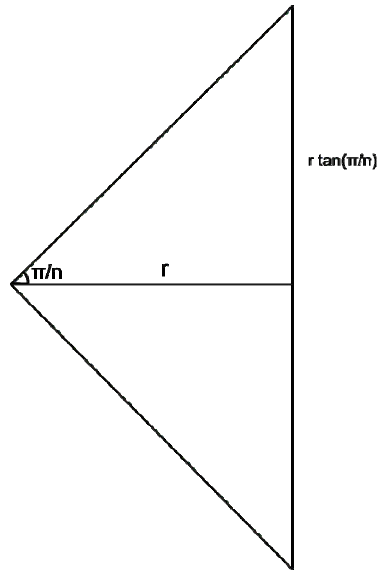
Now, Archimedes accomplished amazing feats by getting up to a regular 96-gon, and estimating square roots, etc. I'm just interested in a nice closed form solution. Here we go in that attempt.

First, note that any regular n-gon can be broken into n isosceles triangles, with each triangle having a vertex at the center of the circle/n-gon and the two adjacent vertices of the n-gon.

This results in our two cases as follows. For the inscribed n-gon, the other vertices lie on the circle, so we get triangles that look like this:



In the circumscribed case, the other vertices lie outside the circle (though each face's midpoint touches the circle) so we get the following triangles:



Note that both of these triangles decompose into two (identical) right triangles, each with a base of one half the full triangle's base. A quick application of trig gives the base of the full inscribed triangle (where r is the radius of the circle, and n is the number of sides in the n-gon) as $2r\sin\left(\frac{\pi}{n}\right)$ and the circumscribed full triangle's base is $2r\tan\left(\frac{\pi}{n}\right)$.

The n triangles partition the n-gon, so we can calculate the circumference of the n-gon by just calculating the base of one of these triangles, and multiplying it by n (the number of triangles).

Now, in each case, we divide by the diameter, $2r$, and we get our two estimates, which under our assumption (which we'll show in just a moment)

$$n\sin\left(\frac{\pi}{n}\right) < \pi < n\tan\left(\frac{\pi}{n}\right)$$

Both of these sequences were designed to converge to π. Indeed, we have

$$\left|n\tan\left(\frac{\pi}{n}\right)-n\sin\left(\frac{\pi}{n}\right)\right|=\left|n\sin\left(\frac{\pi}{n}\right)\left(\frac{1}{\cos(\pi/n)}-1\right)\right|=\left|\pi\underbrace{\frac{\sin(\pi/n)}{(\pi/n)}}_{\to 1}\underbrace{\left(\frac{1-\cos(\pi/n)}{\cos(\pi/n)}\right)}_{\to 0}\right|\to 0 \text{ as}$$

$n\to\infty$.

We can say more. Examining these as continuous functions of x, we get $f(x)=x\sin\left(\frac{\pi}{x}\right)$ and $g(x)=x\tan\left(\frac{\pi}{x}\right)$. We can consider their derivatives:

$$f'(x) = \sin\left(\frac{\pi}{x}\right) - \frac{\pi}{x}\cos\left(\frac{\pi}{x}\right) = \frac{x\sin\left(\frac{\pi}{x}\right) - \pi\cos\left(\frac{\pi}{x}\right)}{x}$$

and

$$g'(x) = \tan\left(\frac{\pi}{x}\right) - \frac{\pi}{x}\sec^2\left(\frac{\pi}{x}\right) = \frac{x\tan\left(\frac{\pi}{x}\right) - \pi\sec^2\left(\frac{\pi}{x}\right)}{x}.$$

We are only interested in the values $x > 3$ so we are necessarily evaluating the trigonometric functions within the first quadrant. Though not obvious, $f'(x)$ is strictly positive in this region, and $g'(x)$ is strictly negative. Thus, our supposed upper bound is strictly decreasing to π, and our (now verified to be) increasing sequence is strictly increasing to π. Thus we have shown that we have upper and lower bounds as stated above.

So, the existence of these bounds is great, but without the ability to calculate trig functions accurately, it seems problematic to calculate bounds for π, and indeed none of this is new; all of this development is essentially due to Archimedes.

Let's push a bit further:

First, recall that $\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ .Next recall that we can massage the power reduction formulas ($\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$ and $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$) to get half angle formulas. In particular, $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos(\theta)}{2}$ thus $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$ and similarly $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$. Combining these, we find that $\tan\left(\frac{\theta}{2}\right) = \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$. We can now note that we are restricted to the first quadrant, we can discard the negative options.

Given that these are all in terms of $\cos(\theta)$, we're going to need to understand how $\cos(\theta)$ behaves. In order to take full use of this half angle formula, we'll restrict ourselves to angles that are divided by powers of two: Let's let $n = 2^j$. (that is, we'll consider only regular (2^j)-gons).

Let's define $c_j = \cos\left(\frac{\pi}{2^j}\right)$. We know that $c_2 = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $c_3 = \cos\left(\left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right)\right)$. By the half angle formula $c_3 = \cos\left(\left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right)\right) = \sqrt{\frac{1+c_2}{2}}$, and in general

$$c_j = \sqrt{\frac{1 + c_{j-1}}{2}}.$$

This is a recurrence relation that we can run forward (from c_2, which we know). So, we can use this relation to find $\cos\left(\frac{\pi}{2^j}\right)$ for any $j > 1$ by calculating all (j-1) values prior to j.

Now, looking at our formulas for our inscribed (lower bound) estimate for π taken from a (2^j)-gon:

$$a_j = 2^j \sin\left(\frac{\pi}{2^j}\right) = 2^j \sin\left(\left(\frac{1}{2}\right)\left(\frac{\pi}{2^{j-1}}\right)\right)$$

(by the half angle formula) $= 2^j \sqrt{\frac{1-\cos\left(\frac{\pi}{2^{j-1}}\right)}{2}} = 2^j \sqrt{\frac{1-c_{j-1}}{2}}$.

Similarly, we can find an overestimate for π from the circumscribed (2^j)-gon:

$b_j = 2^j \tan\left(\frac{\pi}{2^j}\right)$, then $b_j = 2^j \sqrt{\frac{1-c_{j-1}}{1+c_{j-1}}} = 2^j \sqrt{\frac{2}{1+c_{j-1}} - 1}$.

Thus, we have the ability to calculate the inscribed and circumscribed estimates for π by simply determining what $\cos\left(\frac{\pi}{2^j}\right)$ is equal to.

Summarizing:

$$a_j = 2^j \sin\left(\frac{\pi}{2^j}\right) = 2^j \sqrt{\frac{1-c_{j-1}}{2}} < \pi < b_j = 2^j \tan\left(\frac{\pi}{2^j}\right) = 2^j \sqrt{\frac{2}{1+c_{j-1}} - 1}$$

with

$c_j = \cos\left(\frac{\pi}{2^j}\right)$ $\qquad$ $c_j = \sqrt{\frac{1 + c_{j-1}}{2}}$, which can be run forward starting at $c_2 = \frac{1}{\sqrt{2}}$

We now have a nice way for estimating π, using an inscribed and circumscribed (2^j)-gon. Plugging all this into Mathematica, we find out that a regular (2^{40})-gon (that is, a regular 1-trillianish-gon) gives a less tight bound that one might expect! Our bound guarantees about 22 decimal digits of accuracy for our estimate for π in this case. If we jump to a (2^100)-gon, we get about 59 digits of accuracy. Indeed, the number of digits of accuracy varies roughly linearly with j. And for the fun of it, the estimate from the regular (2^{170})-gon gives us about 100 digits of accuracy:

π is approximately:

3. 1415926535 8979323846 2643383279 5028841971 6939937510 5820974944 5923078164 0628620899 8628034825 342117068