Euler's Surprising Series
Joshua Hill

In 1734 Leonhard Euler (roughly pronounced "Oy-ler") discovered that the series $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$. This is extremely non-obvious, but we have most of the tools required to determine why this is true.

In discussion, we found that $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is convergent. At this point you can see this a few ways:

- This is a p-series, with $p=2>1$, so the series is convergent.
- $\sum_{k=2}^{\infty}\frac{1}{k(k-1)}$ converges (as seen during discussion). Both of these series have positive terms, so we can apply the limit comparison test with these two series:
$$\lim_{k\to\infty}\left[\left(\frac{1}{k(k-1)}\right)\Big/\left(\frac{1}{k^2}\right)\right]=\lim_{k\to\infty}\frac{k^2}{k^2-k}=1\text{, so } \sum_{k=1}^{\infty}\frac{1}{k^2}\text{ converges.}$$
- $\frac{1}{k^2}$ is positive and decreasing as k grows larger, so we can use the Integral Test. Using this test, we find that $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges if and only if $\int_1^{\infty}\frac{1}{x^2}dx$ converges. $\int_1^{\infty}\frac{1}{x^2}dx=-\frac{1}{x}\Big|_1^{\infty}=0+1=1$, so $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges. Further, we know that $1=\int_1^{\infty}\frac{1}{x^2}dx\le\sum_{k=1}^{\infty}\frac{1}{k^2}\le\frac{1}{1}+\int_1^{\infty}\frac{1}{x^2}dx=2$ so $1\le\sum_{k=1}^{\infty}\frac{1}{k^2}\le 2$.
- Each term of $\sum_{k=2}^{\infty}\frac{1}{k(k-1)}$ and $\sum_{k=1}^{\infty}\frac{1}{k^2}$ are positive, so we can use the comparison test. $k(k-1)<k^2$ so $\frac{1}{k(k-1)}>\frac{1}{k^2}$ so $1=\sum_{k=2}^{\infty}\frac{1}{k(k-1)}>\sum_{k=2}^{\infty}\frac{1}{k^2}$ so $\sum_{k=1}^{\infty}\frac{1}{k^2}<2$. So $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is convergent.

Notice that the last two approaches gave us estimates for the infinite series, but we never figured out what the series was actually equal to.

To determine this, we start out looking at an apparently unrelated power series:

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$ (The Maclaurin series for $\sin(x)$). Thus, it makes sense that we might consider

$$f(x) = \frac{\sin(x)}{x} = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots\right) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \dots$$

Now we attempt to develop another way of writing out this same function. For any polynomial $p(x)$ with only real roots $r_1, \cdots, r_m$, we can write our polynomial as a product as follows: $p(x) = a(r_1 - x_1)(r_2 - x)\cdots(r_m - x)$ where a is some constant. As long as none of the roots are zero ($r_k \neq 0$ for all k), we could also write this same statement as $p(x) = ar_1r_2\cdots r_m\left(1 - \frac{x}{r_1}\right)\left(1 - \frac{x}{r_2}\right)\cdots\left(1 - \frac{x}{r_m}\right)$, or $p(x) = c\left(1 - \frac{x}{r_1}\right)\left(1 - \frac{x}{r_2}\right)\cdots\left(1 - \frac{x}{r_m}\right)$ where c is a new (related!) constant. Further, in this last form it is clear that $f(0) = c$, so if $f(0) = 1$ then $p(x) = \left(1 - \frac{x}{r_1}\right)\left(1 - \frac{x}{r_2}\right)\cdots\left(1 - \frac{x}{r_m}\right)$.

One might reasonably ask if one could write infinite series in this same way. If an infinite series has an infinite number of roots, can we write this infinite series as an infinite product? It ends up that the answer is "sometimes" but Euler assumed that the answer was "yes", and indeed in this particular instance you can.

Examining $f(x) = \frac{\sin(x)}{x}$, we see that the roots are πk where k is some non-zero integer, and $f(0) = 1$ (recall that $\lim_{x \to 0} \frac{\sin x}{x} = 1$). This seems to satisfy the conditions above, so we might reasonably think of this function as

$$f(x) = \frac{\sin(x)}{x} = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\cdots$$

Now group adjacent terms into a difference of two squares:

$$f(x) = \left[\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\right]\left[\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\right]\left[\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\right]\cdots$$

$$f(x) = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots$$

We want to think about this infinite product as an infinite series again. We can do this by multiplying out the infinite product.

Now we have two ways of writing the same function, so we can extract some information by equating the two series for this function:

$$\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\ldots$$

The coefficient for each power of x should be the same, as the two series are equal.

To get the x^2 term from the product we can multiply exactly one of the x^2 terms from the product and the rest by 1s in the product. We are left with $-\frac{x^2}{\pi^2}-\frac{x^2}{4\pi^2}-\frac{x^2}{9\pi^2}\cdots=x^2\left(\frac{-1}{\pi^2}+\frac{-1}{4\pi^2}+\frac{-1}{9\pi^2}\cdots\right)$, so the x^2 term from the product is $\frac{-1}{\pi^2}+\frac{-1}{4\pi^2}+\frac{-1}{9\pi^2}\cdots=-\frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}$. The x^2 term in the original sum is more obvious, it's just $-\frac{1}{3!}$. These are equal, as they are both just the x^2 coefficient term of our power series, so $-\frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}=-\frac{1}{3!}$ or alternately $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$.

References:

Journey through Genius: The Great Theorems of Mathematics by William Dunham.