Algebra Exam, Spring 2000[1]

1. (40 pts). True or False questions
   (a) The polynomial $X^8 + 4X + 5$ is reducible over $\mathbb{Q}$.
   (b) Assume $|G| = 28$. List all possible orders of a subgroup of G.
   (c) The ideal $I = (x, y)$ of the polynomial ring $R = F[x, y]$ is a free R-module.
   (d) The Galois group of the polynomial $x^p - 2$ over $\mathbb{Q}$ has order $p(p-1)$, where p is a prime number.
   (e) If α has degree m over F and β has degree n over F, then the extension field $F(\alpha, \beta)$ has degree mn.
   (f) For any group G, the map $\theta(g) = g^2$ from G to itself is a homomorphism.
   (g) There are exactly 6 different possible Jordan forms for a 5×5 complex matrix whose characteristic polynomial is $(t-1)^2(t+2)^3$.
   (h) If R is a PID, then the ring $R[X_1, \cdots, X_n]$ is a UFD for every positive n.
2. (10 pts). Factor the number $6 + 9i$ into Gauss primes in the ring $\mathbb{Z}[i]$.
3. (10 pts). Determine the direct sum structure of the abelian group A generated by $\{x, y, z\}$ with the following three relations:
$$7x + 5y + 2z = 0, 3x + 3y = 0, 13x + 11y + 2z = 0.$$
4. (10 pts). Let G by a group of order n which acts non-trivially on a set S of cardinality r. Show that if $n > r!$, then G has a proper normal subgroup.
5. (10 pts). Let K be the splitting field over $\mathbb{Q}$ of the polynomial $f(x) = (x^2 - 2)(x^2 - 3)$. Determine the Galois group G of $f(x)$ and determine all intermediate fields explicitly.
6. (10 pts). Let R be a ring of characteristic p. Prove that if a is a nilpotent then $1 + a$ is unipotent, that is, some power of $1 + a$ is equal to 1.
7. (10 pts). Let G^* be the group of non-zero complex numbers under multiplication. Let H_n be the subgroup of n-th roots of unity. Show that the quotient group C^*/H_n is isomorphic to $C^\star$

[1] Transcribed by Joshua Hill, 2014-03-31. Providence unknown. It is unclear if this is a UCI qualifying exam.