Algebra Qualifying Exam, UCI
Winter 2003

1. Let $\mathbb{F}_p$ denote the field of p elements. Prove the following.
   (a) The ring $\mathbb{F}_2[x]/(x^3+x+1)$ is a field.
   (b) The ring $\mathbb{F}_3[x]/(x^3+x+1)$ is not a field.

2. Let G be a finite group. A character φ of G is a homomorphism $\varphi : G \to \mathbb{C}^*$. Prove that the following conditions are equivalent.
   (a) Every element of G is conjugate to its inverse.
   (b) Every character of G is real-valued.

3. Let p be a prime, V a vector space of dimension p over $\mathbb{Q}$ and $T : V \to V$ a linear transformation such that $T^p =$ identity. Find all possible rational canonical forms for T and teh characteristic polynomial of each.

4. Let $\mathbb{F}_p$ denote the finite field of p elements, where p is prime. Let $UT(n, \mathbb{F}_p)$ denote the group of $n \times n$ upper triangular matrices over $\mathbb{F}_p$ with each of its diagonal entries being 1. Let G be a p-group of order n. Show that G is isomorphic to a subgroup of $UT(n, \mathbb{F}_p)$. (Hint: show that G is isomorphic to a subgroup of $GL(n, \mathbb{F}_p)$ and then use Sylow's theorem via a counting of the group orders).

5. Let V be a vector space and $T : V \to V$ a linear transformation.
   (a) If $\dim_V < \infty$, prove that T is onto if and only if T is 1-1.
   (b) Show by examples that both implications are false if $\dim_V = \infty$.

6. Let N be a submodule of a module M (over an arbitrary ring R). Prove that N is a direct summand of M if and only if there is an endomorphism $f : M \to M$ such that $f \circ f = f$ and $f(M) = N$.

7. Show that the compact group SU_2 has exactly 7 complex representations of dimension 5 and write down all 7 representations in terms of the irreducible representations of SU_2. (Hint: use the fact that SU_2 has one irreducible representation of degree n for each positive integer n.)

8. Let R be the ring $\mathbb{Z}[\sqrt{-5}]$.
   (a) Show that R is not a UFD.
   (b) Factor the principal ideal (6) into a product of prime ideals in the ring R.

9. Let $\mathbb{F}_q$ be the finite field of q elements with characteristic p. Its non-zero elements form a multiplicative group $\mathbb{F}_q^*$ which is cyclic of order $q-1$.

   (a) Let m be a positive integer. Prove that

   $$\sum_{x \in \mathbb{F}_q} x^m = \begin{cases} -1 & \text{if } (q-1) \mid m \\ 0 & \text{otherwise} \end{cases}$$

   (b) Let $n > d$ be positive integers. Let $f(x_1, \ldots, x_n)$ be a polynomial of total degree d in n-variables with coefficients in $\mathbb{F}_q$. Let $N(f)$ denote the number of solutions of the equation

   $$f(x_1, \ldots, x_n) = 0, \; x_i \in \mathbb{F}_q.$$

   Prove that $N(f)$ is divisible by p.

10. For $R = \mathbb{Z}$, give examples of R-modules M where

    (a) M is torsion-free and no linearly independent subset generates M.

    (b) M is free, $X \subset M$ is maximal linearly independent, but X does not generate M.

11. Let K be the splitting field over $\mathbb{Q}$ of the polynomial

    $$f(x) = (x^2 - 2x - 1)(x^4 - 1).$$

    Determine the Galois group G of $f(x)$ and determine all the intermediate fields explicitly.