ALGEBRA QUALIFYING EXAM

June 17, 2008

Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

Notation: Let $\mathbb{F}_q$ denote the finite field with q elements. Let $\mathbb{Z}$ denote the integers. Let $\mathbb{Q}$ denote the rational numbers. Let $\mathbb{R}$ denote the real numbers.

1. Compute the following.
   (a) Suppose G is a cyclic group of order 20. How many automorphisms does G have?
   (b) How many homomorphisms are there from $\mathbb{Z}$ to the symmetric group S_n on n letters?
   (c) If G is a group, and $g \in G$ is an element of order 25, what is the order of g^{10}?

2. Show that if G is a group of order $2pq$, where p and q are (not necessarily distinct) odd primes, then G is not simple.

3. Factor the polynomial $x^4 + 1 \in F[x]$ and find the splitting field over F if the ground field F is:
   (a) $\mathbb{Q}$
   (b) $\mathbb{F}_2$
   (c) $\mathbb{R}$

4. Let $R = \mathbb{Z}[X, Y]$, the ring of polynomials over $\mathbb{Z}$ in the variables X, Y. For each of the following ideals, determine whether the ideal is prime and whether it is maximal. In each case give a short justification.
   (a) (X, Y)
   (b) $(3X, Y)$
   (c) $(X^2 + 1, Y)$
   (d) $(5, X^2 + 1, Y)$

5. Let K be the splitting field of $X^{49} - 1$ over $\mathbb{Q}$. Determine the number of fields F such that $\mathbb{Q} \subseteq F \subseteq K$.

6. Suppose G is a finite group and $H \neq G$ is a subgroup containing every subgroup $K \neq G$ of G.
   (a) Prove that the order of G is a prime power.
   (b) Prove that if G is abelian then G is cyclic.

7. Find all prime ideals in the ring $\mathbb{Z} \times \mathbb{Z}$.

8. Let S be the set of all 6×6 matrices A with entries in $\mathbb{Q}$ such that the characteristic polynomial of A is $x^6 - x^2$ and the minimal polynomial of A is $x^5 - x$.
   (a) If $A, B \in S$, show that A and B are similar.
   (b) Give an example of an element of S.
   (c) If $A \in S$, what is the dimension of the null space of $(A^2 + 1)^2$?

9. Show that the quaternion group Q_8 of order 8 is not a semidirect product of two proper subgroups.

10. Suppose F is an algebraically closed field. Find all monic separable polynomials $f(x) \in F[x]$ such that the set of zeros of $f(x)$ is closed under multiplication.