

Algebra Qualifying Exam , June 2009 (10 points each problem)

- 1 Let D_{2n} be the dihedral group of order $2n$.
 - (a) Prove that if p is an odd prime, then a Sylow p -subgroup of G is normal and cyclic.
 - (b) Prove that if $2n = 2^\alpha \cdot k$ where k is odd then the number of Sylow 2-subgroups of D_{2n} is k . Describe all these subgroups.
- 2 Let G be a group such that $\text{Aut}(G)$ is cyclic. Show that G is abelian.
- 3 Let \mathbb{Z} be the ring of integers, \mathbb{F}_5 be the field with five elements.
 - (a) Determine whether the rings $\mathbb{F}_5[x]/(x^2 + 1)$ and $\mathbb{F}_5[x]/(x^2 + 2)$ are isomorphic.
 - (b) List all ideals in the ring $\mathbb{Z}[x]/(2, x^3 + 1)$.
- 4 Prove that the Galois group of the polynomial $x^5 - 2$ is isomorphic to the group of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a, b \in \mathbb{F}_5$ and $a \neq 0$.

- 5 Let F be a field of characteristic not dividing n . Show that the matrix equation $XY - YX = I_n$ has no solutions, where X and Y are unknown $n \times n$ matrices with entries in F and I_n is the identity matrix.
- 6 Let T be a linear operator on a finite dimensional vector space V over \mathbb{Q} such that $T^{15} = I$. Assume that both T^3 and T^5 have no non-zero fixed points in V . Show that the dimension of V is divisible by 8.
- 7 Let A be a finite Abelian group, p be a prime dividing $|A|$ and k be largest such that p^k divides $|A|$. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes A$ is isomorphic to the Sylow p -subgroup of A .
- 8 Consider complex representations of the finite group S_4 up to isomorphism.
 - (a) Show that S_4 has exactly two one dimensional complex representations.
 - (b) Prove that its other pairwise non-isomorphic complex representations have dimensions 2, 3, and 3.
- 9 Let R be a commutative local ring with maximal ideal M .
 - (a) Show that if $x \in M$, then $1 - x$ is invertible.
 - (b) Show that if in addition that R is Noetherian and I is an ideal satisfying $I^2 = I$, then $I = 0$.
- 10 Let \mathbb{F}_q be a finite field of q elements. Show that every element $x \in \mathbb{F}_q$ can be written as a sum of two squares in \mathbb{F}_q , that is, $x = y^2 + z^2$ for some $y, z \in \mathbb{F}_q$.