ALGEBRA QUALIFYING EXAM not online
June 13, 2011

Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Each question is worth 10 points. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

Notation: Let $\mathbb{Z}$ denote the integers. Let $\mathbb{Q}$ denote the rational numbers. Let $\mathbb{R}$ denote the real numbers. Let $\mathbb{C}$ denote the complex numbers. Let S_n denote the symmetric group on n letters.

1. Let p be an odd prime. Prove that $\mathbb{Q}(e^{2\pi i/p})$ contains a unique quadratic extension of $\mathbb{Q}$. For which p is this quadratic field contained in $\mathbb{R}$? Justify your answer.
2. Prove that the ideal $(2, 3-\sqrt{-5})$ is a maximal ideal in the ring $\mathbb{Z}[\sqrt{-5}]$.
3. A ring R is called *noetherian* if every strictly increasing chain of ideals $I_1 \subsetneq I_2 \subsetneq \cdots$ must be finite in length. Prove that if R is noetherian, then every ideal is finitely generated. Prove that $\mathbb{Z}$ is noetherian.
4. Let G be the Galois group of the polynomial $x^6 - 27$ over $\mathbb{Q}$. Determine all elements of G by describing their actions on the generators of the splitting field. Also describe G as an abstract group.
5. Prove that if G is a group of order $5 \cdot 7 \cdot 11$, then the center of G has order divisible by 7.
6. Describe all maximal ideals in the ring $\mathbb{Z}[X]/(3X)$.
7. Suppose F is a field of characteristic $p > 0$. Define a function $\phi : F \to F$ by $\phi(x) = x^p$.
   (a) Show that ϕ is a field homomorphism.
   (b) Show that if F is finite, then ϕ is an automorphism.
   (c) Give an example of a field F such that ϕ is not an automorphism.
8. Suppose A is an $n \times n$ matrix over $\mathbb{C}$ with minimal polynomial $(x-\lambda)^n$.
   (a) What is the Jordan form of A?
   (b) What is the Jordan form of A^2 when $\lambda \neq 0$?
   (c) What is the Jordan form of A^2 when $\lambda = 0$?
9. Let $V = \mathbb{C}[S_3]$, the complex group ring of S_3. View V as a representation of S_3, with S_3 acting on V by *conjugation* (not by multiplication).
   (a) Give the character table of S_3 (no proof required).
   (b) What is the character of the representation of S_3 on V?
   (c) Express the character of this representation as a sum of irreducible characters.
10. True/False. Mark each question true or false. It is not necessary to show your work.
    (a) If G is a group, and every finitely generated subgroup of G is cyclic, then G is cyclic.
    (b) If G is a nonabelian group, then the center of G is properly contained in some abelian subgroup of G.
    (c) If two 3×3 complex matrices have the same minimal and characteristic polynomials, then they are similar.
    (d) If F is a finite extension of $\mathbb{Q}$ in $\mathbb{C}$, and $F \not\subset \mathbb{R}$, then $[F : \mathbb{Q}]$ must be even.
    (e) If R is a commutative ring with 1, and R has a unique prime ideal, then R is a field.