Math 2J Key

Final

1. Compute the determinant $\det(A)$ and the inverse matrix A^{-1} for

$$A=\begin{bmatrix}1 & 2 & 1\\ 2 & 0 & -3\\ 0 & 1 & 1\end{bmatrix}$$

$$\det\left(\begin{bmatrix}1 & 2 & 1\\ 2 & 0 & -3\\ 0 & 1 & 1\end{bmatrix}\right)=1\begin{vmatrix}0 & -3\\ 1 & 1\end{vmatrix}-2\begin{vmatrix}2 & -3\\ 0 & 1\end{vmatrix}+1\begin{vmatrix}2 & 0\\ 0 & 1\end{vmatrix}=1(3)-2(2)+1(2)=1$$

$$A^{-1}=\frac{\operatorname{adj}A}{\det A}=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}^T=\begin{bmatrix}\begin{vmatrix}0 & -3\\ 1 & 1\end{vmatrix} & -\begin{vmatrix}2 & -3\\ 0 & 1\end{vmatrix} & \begin{vmatrix}2 & 0\\ 0 & 1\end{vmatrix}\\ -\begin{vmatrix}2 & 1\\ 1 & 1\end{vmatrix} & \begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix} & -\begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix}\\ \begin{vmatrix}2 & 1\\ 0 & -3\end{vmatrix} & -\begin{vmatrix}1 & 1\\ 2 & -3\end{vmatrix} & \begin{vmatrix}1 & 2\\ 2 & 0\end{vmatrix}\end{bmatrix}^T=\begin{bmatrix}3 & -2 & 2\\ -1 & 1 & -1\\ -6 & 5 & -4\end{bmatrix}^T=\begin{bmatrix}3 & -1 & -6\\ -2 & 1 & 5\\ 2 & -1 & -4\end{bmatrix}$$

or alternately:

$$\left[\begin{array}{ccc|ccc}1 & 2 & 1 & 1 & 0 & 0\\ 2 & 0 & -3 & 0 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 1\end{array}\right]\xrightarrow{R_2=R_2-2R_1}\left[\begin{array}{ccc|ccc}1 & 2 & 1 & 1 & 0 & 0\\ 0 & -4 & -5 & -2 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 1\end{array}\right]\xrightarrow{R_2\leftrightarrow R_3}\left[\begin{array}{ccc|ccc}1 & 2 & 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0 & 0 & 1\\ 0 & -4 & -5 & -2 & 1 & 0\end{array}\right]$$

$$\xrightarrow{R_3=R_3+4R_2}\left[\begin{array}{ccc|ccc}1 & 2 & 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0 & 0 & 1\\ 0 & 0 & -1 & -2 & 1 & 4\end{array}\right]\xrightarrow[R_1=R_1+R_3]{R_2=R_2+R_3}\left[\begin{array}{ccc|ccc}1 & 2 & 0 & -1 & 1 & 4\\ 0 & 1 & 0 & -2 & 1 & 5\\ 0 & 0 & -1 & -2 & 1 & 4\end{array}\right]\xrightarrow{R_1=R_1-2R_2}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 3 & -1 & -6\\ 0 & 1 & 0 & -2 & 1 & 5\\ 0 & 0 & -1 & -2 & 1 & 4\end{array}\right]\xrightarrow{R_3=-R_3}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 3 & -1 & -6\\ 0 & 1 & 0 & -2 & 1 & 5\\ 0 & 0 & 1 & 2 & -1 & -4\end{array}\right]$$

so $A^{-1}=\begin{bmatrix}3 & -1 & -6\\ -2 & 1 & 5\\ 2 & -1 & -4\end{bmatrix}$

2. Find the general solution to the following system of equations:

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 5 \\ 2 & 3 & -2 & 2 \end{array}\right] \xrightarrow{R_2=R_2-R_1} \left[\begin{array}{ccc|c} 2 & -3 & 1 & 5 \\ 0 & 6 & -3 & -3 \end{array}\right] \xrightarrow[R_2=\frac{1}{6}R_2]{R_1=\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right] \xrightarrow{R_1=R_1+\frac{3}{2}R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{4} & \frac{7}{4} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right] \xrightarrow[R_2=2R_2]{R_1=4R_1} \left[\begin{array}{ccc|c} 4 & 0 & -1 & 7 \\ 0 & 2 & -1 & -1 \end{array}\right] \xrightarrow{\text{In equation form}} \begin{cases} 4x_1 - x_3 = 7 \\ 2x_2 - x_3 = -1 \end{cases}$$

$$\xrightarrow{\text{solve}} \begin{cases} x_1 = \dfrac{7+x_3}{4} \\ x_2 = \dfrac{-1+x_3}{2} \end{cases}$$

We have three variables but only two independent constraints, so our answer will feature one free variable. Letting α be any real number, we get our solution set:
If Our free variable is $\alpha = x_3$, then our solution set is $\left(\frac{7+\alpha}{4}, \frac{-1+\alpha}{2}, \alpha\right)$

If our free variable is $\alpha = x_2$ then we can solve for x_1 and x_3 in terms of x_2 and we get $\left(\frac{1}{2}\alpha + 2, \alpha, 2\alpha + 1\right)$.

If our free variable is $\alpha = x_1$ then we can solve for x_2 and x_3 in terms of x_1 and we get $(\alpha, 2\alpha - 4, 4\alpha - 7)$.

These look different, but describe the same set of points (a line in $\mathbb{R}^3$).

3. Find the characteristic polynomial, the eigenvalues and the eigenvectors for matrix

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}$$

$$p_A(\lambda) = \det(A - \lambda I) = \begin{bmatrix} 4-\lambda & 1 \\ 3 & 6-\lambda \end{bmatrix} = (4-\lambda)(6-\lambda) - 3 = \lambda^2 - 10\lambda + 21$$

The eigenvalues are the roots of the characteristic polynomial:

$p_A(\lambda) = \lambda^2 - 10\lambda + 21 = (\lambda - 3)(\lambda - 7)$ so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 7$.

The eigenvectors are the solutions to $(A - \lambda I)v = 0$:

$\lambda_1 = 3$:

$(A - 4I)v = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} v = 0$. Solving: $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \xrightarrow{R_2 = R_2 - 3R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so $x_1 + x_2 = 0$ or

$x_1 = -x_2$ so taking $\alpha = x_2$ we have the eigenvector $\begin{bmatrix} -\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$\lambda_1 = 7$:

$(A - 7I)v = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} v = 0$. Solving: $\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \xrightarrow{R_2 = R_2 + R_1} \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}$ so $-3x_1 + x_2 = 0$ or

$3x_1 = x_2$ so taking $\alpha = x_1$ we have the eigenvector $\begin{bmatrix} \alpha \\ 3\alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. So $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and

$v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

4. Find the Taylor polynomial $p_3(x)$ of degree 3 for the function $\ln x$ with the center $c = 1$.

$$f(x) = \ln x, \; f^{'}(x) = \frac{1}{x}, \; f^{"}(x) = -\frac{1}{x^2}, \; f^{'''}(x) = \frac{2}{x^3}$$

$$p_3(x) = f(c) + \frac{f^{'}(c)}{1!}(x-c)^1 + \frac{f^{"}(c)}{2!}(x-c)^2 + \frac{f^{'''}(c)}{3!}(x-c)^3 = (x-1)^1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

5. Find the Taylor polynomial $p_2(x)$ of degree 3 for the function $\arctan x$ with the center $c = 0$.

The question seems to be worded ambiguously, so one of two answers would suffice:

$$f(x) = \arctan x, \; f^{'}(x) = \frac{1}{1+x^2}, \; f^{"}(x) = -\frac{2x}{\left(1+x^2\right)^2}, \; f^{'''}(x) = \frac{2\left(3x^2-1\right)}{\left(1+x^2\right)^3}$$

$$p_3(x) = f(c) + \frac{f^{'}(c)}{1!}(x-c)^1 + \frac{f^{"}(c)}{2!}(x-c)^2 + \frac{f^{'''}(c)}{3!}(x-c)^3 = x^1 - \frac{1}{3}x^3$$

or

$$p_2(x) = f(c) + \frac{f^{'}(c)}{1!}(x-c)^1 + \frac{f^{"}(c)}{2!}(x-c)^2 = x^1$$

A student came up with a wonderful solution that avoids most of the messy derivatives:

$$f(x) = \arctan x \quad f^{'}(x) = \frac{1}{1+x^2}$$

$\int \frac{1}{1+x^2}dx = \arctan(x) + C$. $\frac{1}{1+x^2} = \frac{1}{1-\left(-x^2\right)} = \sum_{j=0}^{\infty}\left(-x^2\right)^j = \sum_{j=0}^{\infty}(-1)^j x^{2j}$ so

$\arctan(x) + C = \int \sum_{j=0}^{\infty}(-1)^j x^{2j}dx = \sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}x^{2j+1}$. To solve for C, we can plug in $x = 0$

at which point we find that $C = 0$, so $\arctan(x) = \sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}x^{2j+1}$. Thus we find that

$p_3(x) = x^1 - \frac{1}{3}x^3$.

6. Does the sequence $\left\{\frac{\cos(\pi n)}{n}\right\}$ converge and if it does what is its limit?

$\cos(\pi n) = \begin{cases} 1 & \text{n is even} \\ -1 & \text{n is odd} \end{cases}$ so $\frac{\cos(\pi n)}{n} = \begin{cases} \frac{1}{n} & \text{n is even} \\ -\frac{1}{n} & \text{n is odd} \end{cases}$ $\lim_{n\to\infty}\frac{1}{n} = 0$ and $\lim_{n\to\infty} -\frac{1}{n} = 0$ so $\lim_{n\to\infty}\frac{\cos(\pi n)}{n} = 0$.

Alternately, you could proceed through the squeeze theorem:

$\frac{\cos(\pi n)}{n} = \frac{(-1)^n}{n}$ and $-\frac{1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n}$. Further $\lim_{n\to\infty}\frac{1}{n} = 0$ and $\lim_{n\to\infty} -\frac{1}{n} = 0$ so by the squeeze theorem, $\lim_{n\to\infty}\frac{(-1)^n}{n} = \lim_{n\to\infty}\frac{\cos(\pi n)}{n} = 0$

7. Does the sequence $\left\{\sqrt{n+1} - \sqrt{n}\right\}$ converge, and if it does what is its limit?

$\sqrt{n+1} - \sqrt{n} = \left(\sqrt{n+1} - \sqrt{n}\right)\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$ so

$\lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right) = \lim_{n\to\infty}\frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$

8. Does the series $\sum_{n=0}^{\infty} \sin n$ converge? Explain your answer.

$\lim_{n\to 0} \sin n \ne 0$ so $\sum_{n=0}^{\infty} \sin n$ diverges.

9. Does the series $\sum_{n=0}^{\infty} \frac{\sin n}{2^n}$ converge? Explain your answer.

$\left|\frac{\sin n}{2^n}\right| = \frac{|\sin n|}{2^n} < \frac{1}{2^n}$. $\sum_{n=0}^{\infty}\frac{1}{2^n}$ converges so $\sum_{n=0}^{\infty}\left|\frac{\sin n}{2^n}\right|$ converges by comparison test. Thus $\sum_{n=0}^{\infty}\frac{\sin n}{2^n}$ converges absolutely.

10. What is the radius of convergence for the power series $f(x)=\sum_{n=0}^{\infty} nx^n$? Explain your answer.

$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)}{n}\right|=1$ so the radius of convergence is $R=\frac{1}{L}=1$.

11. What is the radius of convergence for the power series $f(x)=\sum_{n=0}^{\infty}\frac{(x-1)^n}{\sqrt{n}\,2^n}$? Explain your answer.

Note, the question should likely be $f(x)=\sum_{n=1}^{\infty}\frac{(x-1)^n}{\sqrt{n}\,2^n}$ as the index $n=0$ results in a division by 0.

$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{1}{2^{n+1}\sqrt{n+1}}}{\frac{1}{2^n\sqrt{n}}}\right|=\lim_{n\to\infty}\left|\frac{2^n\sqrt{n}}{2^{n+1}\sqrt{n+1}}\right|=\lim_{n\to\infty}\left|\frac{\sqrt{n}}{2\sqrt{n+1}}\right|=\frac{1}{2}$ so the radius of convergence is $R=2$.

12. Find the Maclaurin series of $\sqrt[3]{1+x^2}$ up to the order 3.

$f(x)=\sqrt[3]{1+x^2}$ $\quad f'(x)=\frac{2x}{3(1+x^2)^{\frac{2}{3}}}$ $\quad f''(x)=-\frac{2(x^2-3)}{9(x^2+1)^{\frac{5}{3}}}$ $\quad f'''(x)=\frac{8x(x^2-9)}{27(x^2+1)^{\frac{8}{3}}}$

$p_3(x)=f(0)+\frac{f'(0)}{1!}(x-0)^1+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3=1+\frac{1}{3}x^2$

13. Does the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}$ converge? Explain your answer.

This is a geometric series with the common ratio $r=-\frac{1}{2}$. $|r|<1$ so this series converges to $\frac{1}{1-r}=\frac{1}{1+\frac{1}{2}}=\frac{2}{3}$.

14. Find the Maclaurin series of $\ln(1+x)$.

$f(x)=\ln(1+x)$, $f'(x)=\dfrac{1}{1+x}$, $f''(x)=-\dfrac{1}{(1+x)^2}$, $f'''(x)=\dfrac{2}{(1+x)^3}$,

$f''''(x)=-\dfrac{6}{(1+x)^4}$ so following the pattern, $f^{(n)}(x)=\dfrac{(-1)^{n+1}(n-1)!}{(1+x)^n}$ for $n\geq 0$.

$f(0)=\ln(1)=0$ so we can start the series at $n=1$:

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n=\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n-1)!}{n!}x^n=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$$

You could also do the geometric series trick described above here:

$f(x)=\ln(1+x)$, $f'(x)=\dfrac{1}{1+x}$ so $f'(x)=\dfrac{1}{1-(-x)}=\displaystyle\sum_{j=0}^{\infty}(-x)^j=\sum_{j=0}^{\infty}(-1)^j x^j$ so

$\ln(1+x)+C=\displaystyle\int\sum_{j=0}^{\infty}(-1)^j x^j dx=\sum_{j=0}^{\infty}\frac{(-1)^j}{j+1}x^{j+1}$. Setting $x=0$ we can solve for C and find that $C=0$ so $\ln(1+x)=\displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j}{j+1}x^{j+1}$. Reindexing, we get

$\ln(1+x)=\displaystyle\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}x^j$ which is equivalent to the series we got using the other method.