Math 2J Key
Test 1

1. Solve the linear system of equations using the row echelon form

$$\begin{cases} x_1 + 5x_2 = 7 \\ x_1 - 2x_2 = -2 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 5 & 7 \\ 1 & -2 & -2 \end{array}\right] \xrightarrow{R_2' = R_2 - R_1} \left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & -7 & -9 \end{array}\right] \xrightarrow{R_2'' = -\frac{1}{7}R_2'} \left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 1 & \frac{9}{7} \end{array}\right] \xrightarrow{R_1''' = R_1'' - 5R_2''} \left[\begin{array}{cc|c} 1 & 0 & \frac{4}{7} \\ 0 & 1 & \frac{9}{7} \end{array}\right]$$

so $x_1 = \frac{4}{7}$ and $x_2 = \frac{9}{7}$.

Thus our answer is the point $\left(\frac{4}{7}, \frac{9}{7}\right)$.

2. Solve the linear system of equations using the row echelon form

$$\begin{cases} x_1 - 3x_3 = 8 \\ 2x_1 + 2x_2 + 9x_3 = 7 \\ x_2 + 5x_3 = -2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right] \xrightarrow{R_2' = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{array}\right] \xrightarrow{R_3'' = R_3' - \frac{1}{2}R_2'} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 0 & -\frac{5}{2} & \frac{5}{2} \end{array}\right]$$

$$\xrightarrow[R_3''' = -\frac{2}{5}R_3'']{R_2''' = \frac{1}{2}R_2''} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & \frac{15}{2} & -\frac{9}{2} \\ 0 & 0 & 1 & -1 \end{array}\right]$$

By the third row, $x_3 = -1$.

By the second row, $x_2 + \frac{15}{2}x_3 = -\frac{9}{2}$ or $x_2 - \frac{15}{2} = -\frac{9}{2}$ so $x_2 = 3$.

By the first row, $x_1 - 3x_3 = 8$ so $x_1 + 3 = 8$ so $x_1 = 5$

Thus our answer is the point $(5, 3, -1)$

3. Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & -1 & 3 \\ 0 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix}$$

It's unfun to calculate this directly, so we'll apply Gaussian elimination:

$$\underbrace{\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & -1 & 3 \\ 0 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix}}_{A} \xrightarrow[R_4^{'}=R_4-2R_1]{R_2^{'}=R_2-2R_1} \underbrace{\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -4 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -3 & 1 & -1 \end{bmatrix}}_{A^{'}} \xrightarrow{R_2^{'} \leftrightarrow R_3^{'}} \underbrace{\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -4 & -1 & 1 \\ 0 & -3 & 1 & -1 \end{bmatrix}}_{A^{''}}$$

$$\xrightarrow[R_4^{'''}=R_4^{''}+3R_1^{''}]{R_3^{'''}=R_3^{''}+4R_2^{''}} \underbrace{\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 7 & 2 \end{bmatrix}}_{A^{'''}} \xrightarrow{R_4^{''''}=R_4^{'''}-R_3^{'''}} \underbrace{\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & -3 \end{bmatrix}}_{A^{''''}}$$

Based on the operations that we are performing, we get $\det A = \det A^{'} = -\det A^{''} = -\det A^{'''} = -\det A^{''''}$, so we can calculate any of these. One good place to stop is $A^{'}$ where you can calculate the determinant of a 3×3 by expanding down the first row of $A^{'}$:

$$\det A^{'} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -4 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -3 & 1 & -1 \end{bmatrix} = 1 \begin{vmatrix} -4 & -1 & 1 \\ 1 & 2 & 1 \\ -3 & 1 & -1 \end{vmatrix} = (-4)\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix} + (1)\begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix}$$

$$= (-4)(-3) - (-1)(2) + (1)(7) = 12 + 2 + 7 = 21$$

Another reasonable place to stop would be $A^{''''}$ (which is a triangular matrix), so its determinant is just the product of its diagonal entries:

$$\det A = -\det A^{''''} = -\det \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & -3 \end{bmatrix} = -(1)(1)(7)(-3) = 21$$

4. Find the adjoint and inverse matrices for the given matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$$

The adjoint is defined in terms of cofactors: $A_{ij} = (-1)^{i+j} \det M_{ij}$

$$\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$$

$A_{11} = -1 \qquad A_{12} = -3$

$A_{21} = -2 \qquad A_{22} = 1$ so $\operatorname{adj} A = \begin{bmatrix} -1 & -2 \\ -3 & 1 \end{bmatrix}$

To calculate the inverse, the easy way to proceed is to remember that $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$.

$\det A = -1 - 6 = -7$

$$A^{-1} = \frac{1}{-7} \begin{bmatrix} -1 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{3}{7} & -\frac{1}{7} \end{bmatrix}$$

5. Find the adjoint and inverse matrices for the given matrix:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ -2 & -1 & 1 \end{bmatrix}$$

The adjoint is defined in terms of cofactors: $A_{ij} = (-1)^{i+j} \det M_{ij}$

$$\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$A_{11} = \begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} = 0 \qquad A_{12} = -\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = 3 \qquad A_{13} = \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} = 3$$

$$A_{21} = -\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = -4 \qquad A_{22} = \begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} = 8 \qquad A_{23} = -\begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} = 0$$

$$A_{31} = \begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} = -8 \qquad A_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = 7 \qquad A_{33} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

so $\operatorname{adj} A = \begin{bmatrix} 0 & -4 & -8 \\ 3 & 8 & 7 \\ 3 & 0 & 3 \end{bmatrix}$

To calculate the inverse, the easy way to proceed is to remember that $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$.

$$\det A = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ -2 & -1 & 1 \end{vmatrix} = (2)\begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} - (1)\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + (3)\begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} = 0 + 3 + 9 = 12$$

$$A^{-1} = \frac{1}{12}\begin{bmatrix} 0 & -4 & -8 \\ 3 & 8 & 7 \\ 3 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{4} & \frac{2}{3} & \frac{7}{12} \\ \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix}$$