Math 2J Key
Quiz 2

1. $A=\begin{bmatrix} c & 0 & c \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix}$. What values for c make A invertible?

A is invertible if and only if $\det A \neq 0$. Expanding the determinant along the top row, we see:

$$\det A = c\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} - 0\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} + c\begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix} = c \text{ so } A \text{ is invertible if and only if } c \neq 0.$$

It was not sufficient to note that when $c=0$ the top row was all zeros: this would certainly be an example where A was not invertible, but it does not justify why only this value of c makes A non-invertible.

2. If $c=1$, use Gaussian elimination to find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 3 & 3 & 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3'=R_3-2R_1]{R_2'=R_2-3R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & -3 & 1 & 0 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3''=R_3'-\frac{2}{3}R_2'} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & -3 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & -\frac{2}{3} & 1 \end{array}\right] \xrightarrow{R_3'''=3R_3''} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 3 \end{array}\right]$$

$$\xrightarrow[R_2''''=R_2'''-R_3''']{R_1''''=R_1'''-R_3'''} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -3 \\ 0 & 3 & 0 & -3 & 3 & -3 \\ 0 & 0 & 1 & 0 & -2 & 3 \end{array}\right] \xrightarrow{R_2'''''=\frac{1}{3}R_2''''} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -3 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -2 & 3 \end{array}\right]$$

so $A^{-1}=\begin{bmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix}$

3. What is the determinant of A if $c = 1$? (Hint: you may be able to reuse some prior work!)
By 1, we know that $\det A = c = 1$.

Alternately, note that the matrix after the second round of elimination is this: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$.
The elimination steps up to this point don't modify the determinant. This intermediate matrix is an upper triangular matrix, so its determinant is the diagonal entries multiplied together, so $\det A = (1)\left(\frac{1}{3}\right)(3) = 1$.

4. Use your answer from #2 to solve the following system of equations:

$$\begin{array}{ccccccc} x_1 & & & + & x_3 & = & 1 \\ 3x_1 & + & 3x_2 & + & 4x_3 & = & 2 \\ 2x_1 & + & 2x_2 & + & 3x_3 & = & 1 \end{array}$$

This system of equations is the same as $\underbrace{\begin{bmatrix} c & 0 & c \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{\mathbf{b}}$ or $A\mathbf{x} = \mathbf{b}$. If we left multiply both sides by A^{-1}, we get $A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$, which is the same as $\mathbf{x} = A^{-1}\mathbf{b}$, so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}.$$