Math 2J Key
Quiz 3

1. Diagonalize the following matrices (over the real numbers) if possible. If the matrix is diagonalizable, calculate A^{100}.

a. $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

First, we find the eigenvalues: we see that $p_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 2 \\ -2 & -\lambda \end{bmatrix} = \lambda^2 + 4$.

The roots of the characteristic polynomial are complex, so this matrix is not diagonalizable over the real numbers.

You were not required to calculate A^{100}, but as an aside:

$A = 2\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ so $A^{100} = 2^{100}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{100}$. Let $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Examine a few powers:

$B^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\qquad B^4 = B^2 B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

So, $A^{100} = 2^{100} B^{100} = 2^{100}\left(B^4\right)^{25} = 2^{100} I^{25} = 2^{100} I = \begin{bmatrix} 2^{100} & 0 \\ 0 & 2^{100} \end{bmatrix}$.

b. $A = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}$

To diagonalize, we need the eigenvalues and eigenvectors.

An eigenvector is a vector $\mathbf{v}$ that satisfies the equation $A\mathbf{v} = \lambda\mathbf{v}$ where $\mathbf{v} \neq \mathbf{0}$ for some scalar λ, the eigenvalue. To find these we examine the related system $(A - \lambda I)\mathbf{v} = \mathbf{0}$. As we need non-zero vectors that satisfy this equation, we know the matrix $(A - \lambda I)$ must not be invertible (this was one of our characterizations for a singular matrix!).

We can find out which values of λ cause this to happen (the eigenvalues of A) by finding the roots of the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$. (i.e., we are looking for values of λ that make the matrix $(A - \lambda I)$ singular).

Once we know the eigenvalues, we can find the eigenvectors by solving the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ for each of the eigenvalues.

First, we find the characteristic polynomial:

$$p_A(\lambda) = \det(A-\lambda I) = \det\begin{bmatrix} -3-\lambda & -2 \\ 4 & 3-\lambda \end{bmatrix}$$
$$= (-3-\lambda)(3-\lambda)+8 = -9+3\lambda-3\lambda+\lambda^2+8 = \lambda^2-1$$

The roots of the characteristic polynomial are the eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = -1$.

Solving for the eigenvectors:
For $\lambda_1 = 1$:
$(A-\lambda_1 I)\mathbf{v} = \mathbf{0} = \begin{bmatrix} -4 & -2 \\ 4 & 2 \end{bmatrix}\mathbf{v} = \mathbf{0}$. By Gaussian elimination,
$\begin{bmatrix} -4 & -2 \\ 4 & 2 \end{bmatrix} \xrightarrow{R_2=R_2+R_1} \begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix}$ so in equation form our only constraint is $-4x_1-2x_2=0$ or $x_2=-2x_1$. We'll take x_1 as the free variable. If α is any real number our solution set is $\begin{bmatrix} \alpha \\ -2\alpha \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, so the eigenvector corresponding with $\lambda_1 = 1$ is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

For $\lambda_2 = -1$:
$(A-\lambda_2 I)\mathbf{v} = \mathbf{0} = \begin{bmatrix} -2 & -2 \\ 4 & 4 \end{bmatrix}\mathbf{v} = \mathbf{0}$. By Gaussian elimination,
$\begin{bmatrix} -2 & -2 \\ 4 & 4 \end{bmatrix} \xrightarrow{R_2=R_2+2R_1} \begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix}$ so in equation form our only constraint is $-2x_1-2x_2=0$ or $x_2=-x_1$. We'll take x_1 as the free variable. If α is any real number our solution set is $\begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so the eigenvector corresponding with $\lambda_2 = -1$ is $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

To diagonalize:
$X = [\mathbf{v}_1 \quad \mathbf{v}_2] = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$, $X^{-1} = \dfrac{\operatorname{adj} X}{\det X} = \dfrac{1}{1}\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$. By our construction we know that $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Alternately, we could solve for D:
$$D = X^{-1}AX = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

So $A = XDX^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$.

Calculations of the form A^{100} were discussed a few times. For a reminder, see the comment in your book following theorem 6.3.2, your discussion notes, or homework section 6.3 #2. To summarize:

Powers of A are easy when represented in this diagonalized form. To see why, examine $A^2 = \left(XDX^{-1}\right)\left(XDX^{-1}\right) = XD\left(X^{-1}X\right)DX^{-1} = XD^2X^{-1}$. The same grouping works for any positive integer power, so $A^n = XD^nX^{-1}$. Diagonal matrices are easy to multiply; multiplying out a few terms should convince you that $D^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$.

$$A^{100} = XD^{100}X^{-1} = X\begin{bmatrix} 1^{100} & 0 \\ 0 & -1^{100} \end{bmatrix}X^{-1} = XIX^{-1} = XX^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. Find the 3rd Taylor Polynomial $p_3(x)$ expanded about $c = 1$ for the following:

$f(x) = \sin(x)$

$$p_3(x) = f(c) + \frac{f^{(1)}(c)}{1!}(x-c)^1 + \frac{f^{(2)}(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3$$

$f^{(1)}(x) = \cos(x)$, $f^{(2)}(x) = -\sin(x)$, $f^{(3)}(x) = -\cos(x)$ so

$$p_3(x) = \sin(1) + \cos(1)(x-1) - \frac{\sin(1)}{2}(x-1)^2 - \frac{\cos(1)}{6}(x-1)^3.$$